



MACHAKOS UNIVERSITY

University Examinations 2018/2019

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE

FIRST YEAR SECOND SEMESTER EXAMINATIONS FOR
BACHELOR OF SCIENCE (MATHEMATICS AND STATISTICS)

SAC 102: MATHEMATICAL MODELING

DATE: 10/5/2019

TIME: 8:30 – 10:30 AM

INSTRUCTION:

Answer Question ONE which is compulsory and any other TWO Questions

QUESTION ONE (COMPULSORY) (30 MARKS)

- a) Define the following terms:
- i) A differential Equation. (1 mark)
 - ii) Half-life. (1 mark)
 - iii) Doubling time. (1 mark)
 - iv) Equilibrium state. (1 mark)
 - v) Interactions. (1 mark)
- b) Explain two types of growth rates. (2 marks)
- c) Find both equilibrium solutions of $\frac{ds}{dt} = 2s^2 + s - 15$ and determine their stability. (4 marks)
- d) Find the particular solution to $\frac{dX}{dt} = \frac{3X + 5}{2}$ satisfying $X(0) = 7$. (4 marks)
- e) Verify that a fixed point of $Y_{n+1} = \frac{3Y_n^2 - 6Y_n + 8}{4}$ is $Y_{eq} = 2$ and find the other fixed point. Determine the stability of both fixed points. (6 marks)
- f) List three applications of Logistic Models. (3 marks)
- g) Solve $Q_{n+2} = 5Q_{n+1} - 6Q_n$ with $Q_0 = 0, Q_1 = 2$. (6 marks)

QUESTION TWO (20 MARKS)

- a) The interaction between a host species and a parasite species is modelled by the following pair of differential equations

$$P'(t) = 7P(t) - 4Q(t)$$

$$Q'(t) = 5P(t) - 2Q(t)$$

Where $P(t)$ and $Q(t)$ give the populations of the host and parasites at time t respectively. The time is measured in years.

- i) Find the general solutions for $P(t)$ and $Q(t)$. (4 marks)
- ii) Find the particular solution for $P(0) = 1500$ and $Q(0) = 2000$ and use it to determine the long term behavior of both species. (4 marks)
- iii) Assume that $P(0)$ and $Q(0)$ are both positive and let R represent the average number of parasites per host when $t = 0$. For which range of values of R do both species survive indefinitely. (3 marks)
- b) Determine the equilibrium and stability of $\frac{dp}{dt} = \cos Q$. (2 marks)

Hence:

- i) Sketch the growth rate curve (Phase diagram), (4 marks)
- ii) Sketch the general solution. (3 marks)

QUESTION THREE (20 MARKS)

- a) Consider the prey predator system below, where x represents the prey and y the predators.

i.
$$\frac{dx}{dt} = 3x - 9xy$$
$$\frac{dy}{dt} = -y + 4xy$$

ii.
$$\frac{dx}{dt} = 7x - \frac{1}{3}xy$$
$$\frac{dy}{dt} = -y + \frac{1}{4}xy$$

- i) In which system does the prey reproduce more quickly when there are no predators (justify your answer)? (2 marks)
- ii) In which system are the predators more successful at catching prey (justify your answer)? (2 marks)

- iii) Modify the first model (i.) in such a way that it includes the effect of hunting of predator at a rate $\alpha = 0.2$ proportional to the number of predator. (1 mark)
- iv) Suppose in the second model (ii.) that in the absence of predator, the prey population grows logistically with a carrying capacity of 80. Write the system which takes into account the above assumption. (2 marks)
- b) A public health campaign causes the contagiousness of a disease to decay exponentially. The spread of the epidemic can now be modelled by the equation $\frac{dp}{dt} = re^{-at} p(1-p)$ where p represents the fraction of the population with the disease, r and a are both positive constants and t is the time in days.
- i) Use separation of variables and partial fractions to determine the general solution to the above differential equation. (5 marks)
- ii) If $r = 0.2$, $a = 0.04$ and $p(0) = 0.1$ calculate the time t when half of the population is infected. (3 marks)
- iii) Using the same parameters given above, calculate the fraction of the population that is infected when $t = 40$. (3 marks)
- iv) Let $R(t) = \frac{p(t)}{1-p(t)}$ be the ratio of infected to uninfected individuals. Use the general solution to show that $\lim_{t \rightarrow \infty} R(t) = R(0)e^{r/a}$. (2 marks)

QUESTION FOUR (20 MARKS)

- a) Consider the one parameter family model described by the equation $\frac{dy}{dt} = y^3 + \alpha y + y$
- i) Locate the bifurcation value and describe the bifurcation that takes place. (4 marks)
- ii) Draw the bifurcation diagram. (3 marks)
- b) Consider a population described by the differential equation $\frac{dy}{dt} = y^2 - 4y + 2$.
- i) Find the equilibrium points, their stability and draw the phase diagram. (4 marks)
- ii) Describe the long-term behavior of the population with the given initial population:
- a) $y(0) = 2$ (2 marks)
- b) $y(0) = 5$ (2 marks)

- c) Find the particular solution to $\frac{dx}{dt} = 0.1x(4-x)$, where $x(0) = 5$. (5 marks)

QUESTION FIVE (20 MARKS)

- a) The interaction of two biochemical reagents is monitored at regular intervals and can be modelled by the following pair of coupled recurrence relations

$$X_{n+1} = \frac{6}{5}X_n + \frac{7}{10}Y_n$$

$$Y_{n+1} = \frac{3}{10}X_n + \frac{4}{5}Y_n$$

Where X_n and Y_n denote the amount of each reagent after n observations.

- i) Show that the characteristic equation is $r^2 - 2r + \frac{3}{4} = 0$. (2 marks)
- ii) Find the general solution for X_n and Y_n . (4 marks)
- iii) Find the particular solution when $X_0 = 900$ and $Y_0 = 100$. (3 marks)
- iv) Show that as $n \rightarrow \infty$ the ratio X_n/Y_n tends to $7/3$. (1 mark)
- v) Show that the limit in the answer to part (d) does not depend on the initial amounts. (2 marks)
- b) Given $X_{n+1} = \frac{1}{6}X_n^2(5 - X_n)$. Find:
- i) Fixed points. (2 marks)
- ii) Equilibrium condition and stability. (3 marks)
- iii) Approximate general solution. (3 marks)