MACHAKOS UNIVERSITY

UNIVERSITY EXAMINATIONS 2018/2019

FIRST YEAR SECOND SEMESTER EXAMINATIONS FOR THE DEGREE OF BACHELOR OF SCIENCE IN MATHEMATICS AND STATISTICS

SAC 102: MATHEMATICAL MODELLING

DATE : TIME: 2 HOURS

INSTRUCTIONS: Attempt Question ONE and any other TWO questions.

QUESTION ONE (30 MARKS)

a. Define the following terms:
   i. A differential Equation. (1 mark)
   ii. Half-life. (1 mark)
   iii. Doubling time. (1 mark)
   iv. Equilibrium state. (1 mark)
   v. Interactions. (1 mark)

b. Explain two types of growth rates. (2 marks)

c. Find both equilibrium solutions of $\frac{ds}{dt} = 2s^2 + s - 15$ and determine their stability.

(4 marks)

d. Find the particular solution to $\frac{dX}{dt} = \frac{3X+5}{2}$ satisfying $X(0) = 7$. (4 marks)

e. Verify that a fixed point of $Y_{n+1} = \frac{3Y_n^2 - 6Y_n + 8}{4}$ is $Y_{eq} = 2$ and find the other fixed point. Determine the stability of both fixed points. (6 marks)

f. List three applications of Logistic Models. (3 marks)

g. Solve $Q_{n+2} = 5Q_{n+1} - 6Q_{n+1}$ with $Q_0 = 0, Q_1 = 2$. (6 marks)

QUESTION TWO (20 MARKS)

1. The interaction between a host species and a parasite species is modelled by the following pair of differential equations

$$P^{'}(t) = 7P(t) - 4Q(t)$$
$$Q^{'}(t) = 5P(t) - 2Q(t)$$

Where $P(t)$ and $Q(t)$ give the populations of the host and parasites at time t respectively. The time is measured in years.

i. Find the general solutions for $P(t)$ and $Q(t)$. (4 marks)

ii. Find the particular solution for $P(0) = 1500$ and $Q(0) = 2000$ and use it to determine the long term behavior of both species. (4 marks)

iii. Assume that $P(0)$ and $Q(0)$ are both positive and let R represent the average number of parasites per host when $t = 0$. For which range of values of R do both species survive indefinitely. (3 marks)

2. Determine the equilibrium and stability of $\frac{dp}{dt} = \cos\theta$. (2 marks)

Hence:

i. Sketch the growth rate curve (Phase diagram), (4 marks)

ii. Sketch the general solution. (3 marks)

QUESTION THREE (20 MARKS)

1. Consider the prey predator system below, where x represents the prey and y the predators.

   i. $\frac{dx}{dt} = 3x - 9xy$, $\frac{dy}{dt} = -y + 4xy$

   ii. $\frac{dx}{dt} = 7x - \frac{1}{3}xy$, $\frac{dy}{dt} = -y + \frac{1}{4}xy$

   a. In which system does the prey reproduce more quickly when there are no predators (justify your answer)? (2 marks)
   b. In which system are the predators more successful at catching prey (justify your answer)? (2 marks)
   c. Modify the first model $(i.)$ in such a way that it includes the effect of hunting of predator at a rate $\alpha = 0.2$ proportional to the number of predator. (1 mark)
   d. Suppose in the second model $(ii.)$ that in the absence of predator, the prey population grows logistically with a carrying capacity of 80. Write the system which takes into account the above assumption. (2 marks)

2. A public health campaign causes the contagiousness of a disease to decay exponentially. The spread of the epidemic can now be modelled by the equation $\frac{dp}{dt} = re^{-at}p(1-p)$ where p represents the fraction of the population with the disease, r and a are both positive constants and t is the time in days.

   a. Use separation of variables and partial fractions to determine the general solution to the above differential equation. (5 marks)
   b. If $r = 0.2$, $a = 0.04$ and $p(0) = 0.1$ calculate the time t when half of the population is infected. (3 marks)
   c. Using the same parameters given above, calculate the fraction of the population that is infected when $t = 40$. (3 marks)

d. Let $R(t)=\dfrac{p(t)}{1-p(t)}$ be the ratio of infected to uninfected individuals. Use the general solution to show that $\lim_{t\to\infty} R(t)=R(0)e^{r/a}$. (2 marks)

QUESTION FOUR (20 MARKS)

1. Consider the one parameter family model described by the equation

$$\frac{dy}{dt}=y^3+\alpha y+y$$

a. Locate the bifurcation value and describe the bifurcation that takes place. (4 marks)

b. Draw the bifurcation diagram. (3 marks)

2. Consider a population described by the differential equation $\dfrac{dy}{dt}=y^2-4y+2$.

a. Find the equilibrium points, their stability and draw the phase diagram. (4 marks)

b. Describe the long-term behavior of the population with the given initial population:

i. $y(0)=2$ (2 marks)

ii. $y(0)=5$ (2 marks)

3. Find the particular solution to $\dfrac{dx}{dt}=0.1x(4-x)$, where $x(0)=5$. (5 marks)

QUESTION FIVE (20 MARKS)

1. The interaction of two biochemical reagents is monitored at regular intervals and can be modelled by the following pair of coupled recurrence relations

$$X_{n+1}=\frac{6}{5}X_n+\frac{7}{10}Y_n$$
$$Y_{n+1}=\frac{3}{10}X_n+\frac{4}{5}Y_n$$

Where X_n and Y_n denote the amount of each reagent after n observations.

a. Show that the characteristic equation is $r^2 - 2r + \frac{3}{4} = 0$. (2 marks)

b. Find the general solution for X_n and Y_n. (4 marks)

c. Find the particular solution when $X_0 = 900$ and $Y_0 = 100$. (3 marks)

d. Show that as $n \to \infty$ the ratio X_n / Y_n tends to $7/3$. (1 mark)

e. Show that the limit in the answer to part (d) does not depend on the initial amounts. (2 marks)

2. Given $X_{n+1} = \frac{1}{6} X_n^{\,2}(5 - X_n)$. Find:

   a. Fixed points. (2 marks)

   b. Equilibrium condition and stability. (3 marks)

   c. Approximate general solution. (3 marks)