MACHAKOS UNIVERSITY

University Examinations 2019/2020 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

SMA 437: BIOMATHEMATICS

DECEMBER, 2019 **TIME: 2 HRS**

INSTRUCTIONS: Answer Question ONE and any other TWO Questions

QUESTION ONE (30MARKS) COMPULSORY

a) Solve the logistic equation $\frac{dN}{dt} = \alpha\left(1 - \frac{N}{M}\right)N$, by making the equation separable.(5 marks)

b) A poultry farmer in lower Eastern Kenya has a farm that can support a 500 chicken, which he sells to fast food cafes in the nearby city. The population of the chicken grows at a rate of 20% per year.

i) What is the maximum number of chicken that can be sold out each year, sustainably? (4 marks)

ii) In order to maintain the flock size at 250 chickens, how many chickens should be sold each year? What should be the minimum size of the initial flock for this to happen? (4 marks)

c) The non-dimensional Lokta-Volterra model is given $\begin{aligned}\dot{u} &= u(1-u)\\ \dot{v} &= \alpha v(u-1)\end{aligned}$ where α is a positive constant. Given that the phase trajectories for this model are $\alpha u + v - \ln(u^{\alpha} v) = C$ where C is a constant. Sketch the phase trajectories on the uv phase plane and briefly explain the dynamics (6 marks)

d) A Volterra model for the population size $p(t)$ of a species is, in reduced form $k\frac{dp}{dt} = p - p^2 - p\int_0^t p(s)ds,\ p(0) = p_0$ where the integral represents a toxicity accumulation term.

i) Letting $x = \ln p$ show that x satisfies $k\ddot{x} + e^x\dot{x} + e^x = 0$ (6 marks)

ii) Use $y = \dot{x}$ to show that the system is also equivalent to $\dot{y} = -\frac{(y+1)p}{k}$ and $\dot{p} = yp$

determine the phase path in terms of y and p (5 marks)

QUESTION TWO (20 MARKS)

a) The spread of common cold/flu in a given confined population can be modeled using the SIR model:

$$\dot{S} = -\beta SI, \quad \beta > 0, S(0) > 0$$
$$\dot{I} = \beta SI - \sigma I, \sigma > 0, I(0) > 0$$
$$\dot{R} = \sigma I, \ R(0) = 0$$

i) Sketch the flow diagram used to deduce the above model equations.(4 marks)

ii) State any three assumptions used to develop the above SIR model that describes the spread of the common cold/flu. (3 marks)

iii) Realistically, the population size changes and common cold is known to recur i.e. a recovered individual can get the disease once again. Improve the given model equations to incorporate this realistic phenomena. (3 marks)

b) The populations x and y of a two-species ecosystem satisfy the equations: $\dot{x} = x - 5y$, $\dot{y} = x - y$

i) Find the equilibrium points of the system. (2 marks)

ii) Using the substitution $y = zx$ show that the phase paths of this system is a family of ellipses given by $x^2 - 2xy + 5y^2 = C$. (8 marks)

QUESTION THREE (20 MARKS)

a) In a given bacterial population of size $N(t)$, the nutrient-depletion can be modeled as $\frac{dN}{dt} = k(C_0 - \alpha N)N$ where α represents the units of nutrient consumed to produce one unit of population increment. C_0 is the initial amount of nutrient available for the population.

i) Show that the model equation can be written in the form $\frac{dN}{\left(1-\frac{N}{B}\right)N} = rdt$ where $r = kC_0$ and $B = \frac{C_0}{\alpha}$ (3 marks)

ii) Solve the equation in i) above and show that for $t \to \infty$ the population approaches B (7 marks)

b) Given the dynamical system: $\begin{aligned}\dot{x} &= y(1-x^2)\\ \dot{y} &= -x(1-y^2)\end{aligned}$

i) Determine the equilibrium points (5 marks)

ii) Show that the phase paths are given by the equation $(1-x^2)(1-y^2) = C$, where C is a constant. (5 marks)

QUESTION FOUR (20 MARKS)

a) Given the predator, $P(t)$ -prey, $N(t)$ model $\begin{aligned}\dot{N} &= N\left[r\left(1-\frac{N}{K}\right)-\frac{kP}{N+D}\right]\\ \dot{P} &= P\left[s\left(1-\frac{hP}{N}\right)\right]\end{aligned}$ where

r, K, k, D, s, h are positive constants.

i) Determine the corresponding non-dimensional model equations which have only the 3 parameters $a = \frac{k}{hr}$, $b = \frac{s}{r}$, $d = \frac{D}{K}$. Take $P(t) = \frac{Kv(\tau)}{h}$, $\tau = rt$ (7 marks)

ii) For $a = 4$, $d = -2$ find the equilibrium points of the non-dimensional model. (6 marks)

b) Consider the predator, $P(t)$ -prey, $N(t)$ model $\begin{aligned}\dot{N} &= N[a-bP]\\ \dot{P} &= P[cN-d]\end{aligned}$ where a, b, c, d are positive constants. Improve this model such that each of the species $P(t)$ and $N(t)$ have logistic growth in the absence of the other and state the meaning of each constant in the improved model. (7 marks)

QUESTION FIVE (20 MARKS)

The SEIR model consists of the following models equations: $\begin{aligned}\dot{S} &= -\beta SI + \lambda - \mu S\\ \dot{E} &= \beta SI - (\mu + k)E\\ \dot{I} &= kE - (\gamma + \mu)I\\ \dot{R} &= \gamma I - \mu R\end{aligned}$

a) Obtain the matrices F and V for this model. (7 marks)

b) Determine the reproduction number R_0 (3 marks)

c) Determine the Endemic Equilibrium (EE) point. (10 marks)