MACHAKOS UNIVERSITY

University Examinations 2020/2021 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FIRST YEAR SECOND SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE IN ACTUARIAL SCIENCES

SAC 102: MATHEMATICAL MODELLING

DATE: 22/6/2021 **TIME: 8.30-10.30 AM**

INSTRUCTIONS:

Answer Question ONE and ANY OTHER TWO

QUESTION ONE (30 MARKS)

a) Explain briefly two objectives of mathematical modelling giving an example (3 marks)

b) Explain briefly two main categories of mathematical models. (4 marks)

c) The number of cells present in a culture is observed to double each day. Measuring time in days and denoting number of cells present at the end of the t^{th} day by n_t ,determine the expression for n_t ,that is valid for all t. Given that $n_0 = 3$, determine the general formula for $n_t$and number of cells on the 15^{th} day. (4 marks)

d) If a sum of money is invested at 9% compounded continuously, using differential equations determine how long it will take for the sum to double. (8 marks)

e) Consider the spreading of a highly communicable disease on an isolated island with population size N. A portion of the population travels abroad and returns to the island infected with the disease. You would like to predict the number of people X who will have been infected by some time *t*. Consider the following model, where *k>0* is constant:

$$\frac{dX}{dt} = kX(N - X)$$

List two major assumptions implicit in the preceding model. (2 marks)

f) A study was made by a retail merchant to determine the relation between weekly advertising expenditure and sales. The following data were recorded ($):

Advertising costs	40	20	25	20	30	50	40	20	50	40	25	50
Sales	385	400	395	365	475	440	490	420	560	525	480	510

Determine the equation of the regression line to predict weekly sales from advertising expenditures. (5 marks)

g) Define the term equilibrium hence determine the equilibrium and stability of the following equation. (4 marks)

$$\frac{dQ}{dt} = F(Q) = e^{-Q} - Q$$

QUESTION TWO (20 MARKS)

a) Solve the difference equation $U_{t+2} - U_{t+1} - U_t$=0 given the initial conditions U_0=0 and U_0 =0 and U_1=1 (10 marks)

b) A logistic population model is used to estimate the population p(t), t years after the initial observation of the population of the town of Nakuru.

Population p(t) is given by:

P(t)=$\frac{50000}{2+3e^{-0.04t}}$

i. Write down the initial and ultimate population sizes. (2 marks)

ii. Calculate, correct to the nearest whole number whole number the rate of growth of the population 10years after the initial observation. (3 marks)

c) Solve the initial value problem

$$\begin{cases} \frac{dx}{dt} = \alpha x \\ x(0) = x_0 \end{cases}$$

where $\alpha = constant.$ (5 marks)

QUESTION THREE (20MARKS)

a) An investor invests an initial amount P_0.He then adds a regular amount at the end of each year.

i. Obtain an expression for how much the sum of money grows to assuming an interest rate of r% p.a. (3 marks)

ii. What will be the amount assuming $P_0 = £10{,}000$, $r = 12\%$ and the regular amount added is £500? (2 marks)

b) A bacteria culture is known to grow at a rate proportional to the amount present. After one hour, 1000 strands of bacteria are observed in the culture and after 4 hours, 3000 strands. Determine

i. An expression for the number of strands of the bacteria present in the culture at time t.

ii. The number of strands of the bacteria originally in the culture. (8 marks)

c) The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country. If after two years the population has doubled and after 3yrs the population is 20,000. Determine the number of people initially in the country. (7 marks)

QUESTION FOUR (20 MARKS)

a) Consider the differential equation $\frac{dy}{dt} = 2 - 3y + y^2$.

i) Determine its equilibrium solutions.

ii) Analyse the solutions obtained in (i) above for stability by sketching the general solution and the phase diagram. (8 marks)

b) Explain briefly one reason why the linear model for population growth is unrealistic. (2 marks)

c) A rumour spread through Machakos University student population of 1000 students at a rate proportional to the product of the number who have heard the rumour and the number who have not. If five student leaders initiated the rumour and 10 students heard the rumour after 1 day.

i) How many students will have heard the rumour after 7 days?

ii) How long will it take for 850 students to hear the rumour? (10 marks)

QUESTION FIVE (20 MARKS)

a) Integrate $\frac{dy}{dx} = 3x^2e^{-y}$ by separating the variables first. (4 marks)

b) Consider the logistic growth model $\frac{dp}{dt} = \alpha p(1 - \frac{p}{N})$. Determine its equilibrium points and by aid of a phase portrait diagram analysis comment on their stability. (6 marks)

c) If production increases by 4% every year and P_n denotes the production level in year n, write down a difference equation satisfied by P_n. If production was 10 million tonnes in 2010, estimate

i) When it will reach 14 million tonnes.

ii) When it was below 6 tonnes. (6 marks)

d) A disease spreads in such a way that 100 individuals become infected during any year and 25% of those infected at the beginning of a year die before the end of the year. Write down a difference equation for I_n, the number of people infected at the end of n years. What happens in the long term? (4 marks)