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Exponentiated Generalized Transformed-Transformer Family of Distributions

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Abstract

Recently, the development of generalized class of distributions has become an issue of interest, to both applied and theoretical statisticians, due to their wider application in different fields of studies. Thus, the current work proposed a new generalized family of distributions called the exponentiated generalized transformed-transformer family. Some members of the new family such as the exponentiated generalized half logistic family was discussed. Statistical measures such as quantile, moment, moment generating function and Shannon entropy for this new class of distributions have been derived.

Mathematics Subject Classification: 62E15; 60E05

Keywords: Exponentiated generalized T-X; quantile; moment; moment generating function; Shannon entropy

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1 Introduction

Myriad of problems arise in different field of studies such as engineering, actuarial science, environmental, biological studies, demography, economics and finance that requires modeling using suitable probability distribution models. However, the data generating process is often characterized with the problems of elongation and asymmetry, which makes it difficult for the classical distributions to provide adequate fit to the real data. In addition, the data sets may exhibit non-monotonic failure rate such as the bathtub, unimodal and modified unimodal failure rate. Hence, it is often necessary to utilize a general model that is likely to include a model suitable for the data as a special case. These have motivated both theoretical and applied statisticians to develop generators for modifying existing statistical distributions to make them more flexible in modeling real data. For this reason, researchers in the field of distribution theory have developed and studied many generalized classes of distributions.

Cordeiro et al. [4] developed the exponentiated generalized class of distributions. Given a random variable X with cumulative distribution function (CDF) $F(x)$, the CDF of the exponentiated generalized class of distributions is defined as

$$G(x) = \left[1 - (1 - F(x))^{\alpha}\right]^{\beta}. \tag{1}$$

Alzaatreh et al. [1] recently proposed a new family of distributions called the transformed-transformer (T-X) family. They used a non-negative continuous random variable T as a generator and defined the CDF of their class of distribution as

$$G(x) = \int_0^{-\log(1-F(x))} r(t)dt = R\left\{-\log(1 - F(x))\right\}, \tag{2}$$

where $r(t)$ is the probability density function (PDF) of the random variable T.

Alzaatreh et al. [1] T-X family of distribution extends the beta-generated family of [7] by replacing the beta random variable with any non-negative continuous random variable T. The corresponding PDF of the CDF defined in equation (2) is given by

$$\mathrm{g}(x) = \frac{f(x)}{1 - F(x)} r\left\{-\log(1 - F(x))\right\}. \tag{3}$$

Alzaghal et al. [2] proposed an extension of the T-X family by introducing a single shape parameter c to make the family of distributions defined by [1] more flexible. Alzaghal et al. [2] called this new family the exponentiated T-X family. The CDF of the exponentiated T-X family is defined as

$$G(x) = \int_0^{-\log(1-F^c(x))} r(t)dt = R\left\{-\log(1-F^c(x))\right\}. \tag{4}$$

The corresponding PDF is given by

$$\mathrm{g}(x) = \frac{cf(x)F^{c-1}(x)}{1-F^c(x)} r\left\{-\log(1-F^c(x))\right\}, c > 0. \tag{5}$$

It is obvious that the upper limits used in the T-X family and the exponentiated T-X family are cumulative hazard functions of certain families of distributions. Thus, new families of the T-X distributions can be defined by employing new cumulative hazard function as an upper limit. In this study, a new T-X family called the exponentiated generalized (EG) T-X family is proposed by using a new upper limit that generalizes that of [1] and [2] to provide greater flexibility in modeling real data.

2 The New Family

Let $r(t)$ and $R(t)$ be the PDF and CDF of a non-negative random variable T with support $[0, \infty)$ respectively. The CDF of the EG T-X family of distributions for a random variable X is defined as

$$G(x) = \int_0^{-\log[1-(1-\bar{F}^d(x))^c]} r(t)dt = R\left\{-\log[1-(1-\bar{F}^d(x))^c]\right\}, \tag{6}$$

where $\bar{F}(x) = 1 - F(x)$ is the survival function of the random variable X and $c > 0, d > 0$ are shape parameters. The corresponding PDF of the new family is obtained by differentiating equation (6) and is given by

$$\mathrm{g}(x) = cd\frac{f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}}{1-(1-\bar{F}^d(x))^c} r\left\{-\log[1-(1-\bar{F}^d(x))^c]\right\}. \tag{7}$$

Employing similar naming convention as T-X distribution, each member of the new family of distribution generated from (7) is named EG T-X distribution.

When the parameter $d = 1$, the PDF in (7) reduces to the PDF in equation (5). In addition, when $c = d = 1$, the PDF in (7) reduces to the PDF in equation (3). The CDF and PDF of the EG T-X distribution can be written as $G(x) = R\left\{-\log[1-(1-\bar{F}^d(x))^c]\right\} = R(H(x))$ and $\mathrm{g}(x) = h(x)r(H(x))$, where $H(x)$ and $h(x)$ are the cumulative hazard and hazard functions of the random variable X with CDF $[1-(1-F(x))^d]^c$ respectively. Thus, the EG T-X distribution can be described as a family of distribution arising from a weighted hazard function. The hazard function of the EG T-X family is given by

$$\tau(x) = \frac{\mathrm{g}(x)}{1-G(x)} = cd\frac{f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}r\left\{-\log[1-(1-\bar{F}^d(x))^c]\right\}}{\left(1-(1-\bar{F}^d(x))^c\right)\left(1-R\left\{-\log[1-(1-\bar{F}^d(x))^c]\right\}\right)}. \tag{8}$$

Lemma 1. *Let T be a random variable with PDF $r(t)$, then the random variable $X = Q_X\left\{1-\left[1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}$, where $Q_X(\cdot) = F^{-1}(\cdot)$ is the quantile function of the random variable X with CDF $F(x)$, follows the EG T-X distribution.*

Proof. Using the fact that $G(x) = R\left\{-\log[1-(1-\bar{F}^d(x))^c]\right\}$ gives the relationship between the random variable T and X as $T = -\log[1-(1-\bar{F}^d(X))^c]$. Thus, solving for X yields $X = Q_X\left\{1-\left[1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}$. □

Lemma 1 makes it easy to simulate the random variable X by first generating random numbers from the distribution of the random variable T and then computing $X = Q_X\left\{1-\left[1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}$, which has the CDF $G(x)$.

3 Some Exponentiated Generalized Transformed-Transformer Families

The EG T-X family can be categorized into two broad sub-families. One sub-family has the same T distribution but different X distributions and the other sub-family has different T distributions but the same X distribution.

Table 1 displays different EG T-X distributions with different T distributions but the same X distribution.

Table 1: EG T-X Families from Different T Distributions

Name	Density $r(t)$	EG T-X Family density g(x)
Exponential	$\lambda e^{-\lambda t}$	$\frac{cd\lambda f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}}{[1-(1-\bar{F}^d(x))^c]^{1-\lambda}}$
Gamma	$\frac{1}{\Gamma(\alpha)\beta^\alpha}t^{\alpha-1}e^{-\frac{t}{\beta}}$	$cd\frac{f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}\left[1-(1-\bar{F}^d(x))^c\right]^{\frac{1}{\beta}-1}}{\Gamma(\alpha)\beta^\alpha\left\{-\log[1-(1-\bar{F}^d(x))^c]\right\}^{1-\alpha}}$
Gompertz	$\theta e^{\gamma t}e^{-\frac{\theta}{\gamma}(e^{\gamma t}-1)}$	$\frac{cd\theta f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}\exp\left(\frac{\theta}{\gamma}\left\{1-\left[1-(1-\bar{F}^d(x))^c\right]^{-\gamma}\right\}\right)}{\left\{1-\left[1-(1-\bar{F}^d(x))^c\right]\right\}^{\gamma+1}}$
Half logistic	$\frac{2\lambda e^{-\lambda t}}{\left(1+e^{-\lambda t}\right)^2}$	$\frac{2cd\lambda f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}\left[1-(1-\bar{F}^d(x))^c\right]^{\lambda-1}}{\left\{1+\left[1-(1-\bar{F}^d(x))^c\right]^{\lambda}\right\}^2}$
Lomax	$\frac{\lambda k}{(1+\lambda t)^{k+1}}$	$\frac{cd\lambda k f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}\left\{1-\lambda\log[1-(1-\bar{F}^d(x))^c]\right\}^{-k-1}}{[1-(1-\bar{F}^d(x))^c]}$
Burr XII	$\frac{\alpha k t^{\alpha-1}}{(1+t^\alpha)^{k+1}}$	$\frac{cd\alpha k f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}\left\{-\log[1-(1-\bar{F}^d(x))^c]\right\}^{\alpha-1}}{[1-(1-\bar{F}^d(x))^c]\left\{1+\left[-\log\left(1-(1-\bar{F}^d(x))^c\right)\right]^{\alpha}\right\}^{k+1}}$
Weibull	$\frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1}e^{-\left(\frac{t}{\gamma}\right)^\alpha}$	$\frac{cd\alpha f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}\exp\left\{-\left[-\log\left(1-(1-\bar{F}^d(x))^c\right)^{\frac{1}{\gamma}}\right]^{\alpha}\right\}}{\gamma[1-(1-\bar{F}^d(x))^c]\left\{-\log[1-(1-\bar{F}^d(X))^c]\right\}^{1-\alpha}}$

3.1 Exponentiated Generalized Half Logistic Family

If the random variable T follows the half logistic distribution with parameter λ, then $r(t) = \frac{2\lambda e^{-\lambda t}}{\left(1+e^{-\lambda t}\right)^2}, t > 0, \lambda > 0$. Using equation (7), the PDF of the exponentiated generalized half logistic (EGHL) family is defined as

$$\text{g(x)} = \frac{2cd\lambda f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}\left[1-(1-\bar{F}^d(x))^c\right]^{\lambda-1}}{\left\{1+\left[1-(1-\bar{F}^d(x))^c\right]^{\lambda}\right\}^2}. \quad (9)$$

Using the CDF of the half logistic distribution, $R(t)=\frac{1-e^{-\lambda t}}{1+e^{-\lambda t}}$ and equation (6), the corresponding CDF of the EGHL family is given by

$$G(x)=\frac{1-\left[1-\left(1-\bar{F}^{d}(x)\right)^{c}\right]^{\lambda}}{1+\left[1-\left(1-\bar{F}^{d}(x)\right)^{c}\right]^{\lambda}}.$$

The EGHL family generalizes all half logistic families of [2] exponentiated T-X family and [1] T-X family. If the random variable X follows a Frétchet distribution with CDF $F(x)=e^{-\left(\frac{a}{x}\right)^{b}}, x>0, a>0, b>0$, then the CDF of the EGHL-Frétchet distribution (EGHLFD) is given by

$$G(x)=\frac{1-\left\{1-\left[1-\left(1-e^{-\left(\frac{a}{x}\right)^{b}}\right)^{d}\right]^{c}\right\}^{\lambda}}{1+\left\{1-\left[1-\left(1-e^{-\left(\frac{a}{x}\right)^{b}}\right)^{d}\right]^{c}\right\}^{\lambda}}. \tag{10}$$

The corresponding PDF of the EGHLFD is obtained by differentiating (10) and is given by

$$\mathrm{g(x)}=\frac{2a^{b}bcd\lambda\left(1-e^{-\left(\frac{a}{x}\right)^{b}}\right)^{d-1}\left[1-\left(1-e^{-\left(\frac{a}{x}\right)^{b}}\right)^{d}\right]^{c-1}\left\{1-\left[1-\left(1-e^{-\left(\frac{a}{x}\right)^{b}}\right)^{d}\right]^{c}\right\}^{\lambda-1}}{x^{b+1}e^{\left(\frac{a}{x}\right)^{b}}\left\{1+\left\{1-\left[1-\left(1-e^{-\left(\frac{a}{x}\right)^{b}}\right)^{d}\right]^{c}\right\}^{\lambda}\right\}^{2}}. \tag{11}$$

Some special cases of the EGHLFD are:

1. When $\lambda=1$, the EGHLFD reduces to EG standardized half logistic Frétchet distribution.

2. When $b=1$, the EGHLFD reduces to EGHL inverse exponential distribution.

3. When $c=d=1$, the EGHLFD reduces to half logistic Frétchet distribution.

4. When $c=d=b=1$, the EGHLFD reduces to half logistic inverse exponential distribution.

The relationship between the EGHLFD and the uniform, Frétchet and half logistic distributions are given by lemma 2.

Lemma 2. *1. If the random variable Y follows the uniform distribution on the interval $(0,1)$, then the random variable*

$$X = a\left\{-\log\left[1-\left(1-\left(1-\left(\frac{Y}{2-Y}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right]\right\}^{-\frac{1}{b}},$$

has EGHLFD with parameters a, b, c, d and λ.

2. *If the random variable Y follows the Frétchet distribution with parameters a and b, then the random variable*

$$X = a\left\{-\log\left[1-\left(1-\left(1-\left(\frac{1-e^{-\left(\frac{a}{x}\right)^b}}{1+e^{-\left(\frac{a}{x}\right)^b}}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right]\right\}^{-\frac{1}{b}},$$

has EGHLFD with parameters a, b, c, d and λ.

3. *If the random variable Y follows the half logistic distribution with parameter λ, then the random variable*

$$X = a\left\{-\log\left[1-\left(1-\left(1-e^{-Y}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right]\right\}^{-\frac{1}{b}},$$

has EGHLFD with parameters a, b, c, d and λ.

Proof. The results follow directly from transformation of random variables. □

4 Statistical Measures

In this section, we discuss statistical measures such as quantile, moment, moment generating function (MGF)and Shannon entropy of the EG T-X family of distributions.

Lemma 3. *The quantile function of the EG T-X family for $p \in (0,1)$ is given by $Q(p) = Q_X\left\{1-\left[1-\left(1-e^{-Q_T(p)}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}$, where $Q_X(\cdot) = F^{-1}(\cdot)$ is the quantile function of the random variable X with CDF $F(x)$ and $Q_T(\cdot) = R^{-1}(\cdot)$ is the quantile function of the random variable T with CDF $R(t)$.*

Proof. Using the CDF of the EG T-X family defined in equation (6), the quantile function is obtained by solving the equation

$$R\left\{-\log\left[1-\left(1-\bar{F}^{d}(Q(p))\right)^{c}\right]\right\}=p,$$

for $Q(p)$. Thus, the proof is complete. □

Corollary 1. *Based on lemma* 3*, the quantile function for the EGHL family is given by,*

$$Q(p)=Q_X\left\{1-\left[1-\left(1-\left(\frac{1-p}{1+p}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}.$$

Proposition 1. *The r^{th} non-central moment of the EG T-X family of distributions is given by*

$$\mu_r^{'}=\sum_{i,k,l,m=0}^{\infty}\sum_{j=0}^{i}\frac{(-1)^{j+k+l+m}\,l^m\,\delta_{r,i}\,\Gamma\left(i+1\right)\Gamma\left(\frac{j}{d}+1\right)\Gamma\left(\frac{k}{c}+1\right)}{j!\,k!\,l!\,m!\,\Gamma\left(i-j+1\right)\Gamma\left(\frac{j}{d}-k+1\right)\Gamma\left(\frac{k}{c}-l+1\right)}E\left(T^{m}\right), \tag{12}$$

where $\delta_{r,i}=(ih_0)^{-1}\sum_{s=1}^{i}\left[s\left(r+1\right)-i\right]h_s\,\delta_{r,i-s}$ with $\delta_{r,0}=h_0^r$, $h_i\,(i=0,1,\ldots)$ are suitably chosen real numbers that depend on the parameters of the $F(x)$ distribution, $E(T^m)$ is the m^{th} moment of the random variable T, $\Gamma(\cdot)$ is the gamma function and $r=1,2,\ldots$.

Proof. From lemma 1, $X=Q_X\left\{1-\left[1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}$, where $Q_X(\cdot)=F^{-1}(\cdot)$ is a quantile function. Thus, $Q_X(\cdot)=F^{-1}(\cdot)$ can be expressed in terms of power series using the following power series expansion of the quantile.

$$Q_X(u)=\sum_{i=0}^{\infty}h_i u^i, \tag{13}$$

where the coefficients are suitably chosen real numbers that depend on the parameters of the $F(x)$ distribution. For a power series raised to a positive integer r (for $r\geq 1$), we have

$$\left(Q_X(u)\right)^r=\left(\sum_{i=0}^{\infty}h_i u^i\right)^r=\sum_{i=0}^{\infty}\delta_{r,\,i}\,u^i, \tag{14}$$

where the coefficients $\delta_{r,\,i}$ (for $i = 1, 2, \ldots$) are determined from the recurrence equation $\delta_{r,i} = (ih_0)^{-1} \sum_{s=1}^{i} [s\,(r+1) - i]\, h_s\, \delta_{r,i-s}$ and $\delta_{r,0} = h_0^r$ [5]. Using equations (13) and (14), the r^{th} non-central moment of the EG T-X family of distributions can be expressed as

$$E\left(X^r\right) = \mu_r^{'} = E\left\{\sum_{i=0}^{\infty} \delta_{r,\,i} \left[1 - \left(1 - \left(1 - e^{-T}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right]^i\right\}. \tag{15}$$

Since $0 < \left(1 - \left(1 - e^{-T}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}} < 1$, for $T \in [0, \infty)$, applying the binomial series expansion

$$(1 - z)^{\eta} = \sum_{j=0}^{\infty} \frac{(-1)^j\, \Gamma\,(\eta + 1)}{j!\,\Gamma\,(\eta - j + 1)} z^{\eta}, \mid z \mid < 1,$$

for real non-integer $\eta > 0$, thrice, we obtain

$$\left[1 - \left(1 - \left(1 - e^{-T}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right]^i = \sum_{k,\,l=0}^{\infty} \sum_{j=0}^{i} \frac{(-1)^{j+k+l}\, \Gamma\,(i+1)\, \Gamma\left(\frac{j}{d} + 1\right) \Gamma\left(\frac{k}{c} + 1\right) e^{-lT}}{j!\,k!\,l!\,\Gamma\,(i - j + 1)\, \Gamma\left(\frac{j}{d} - k + 1\right) \Gamma\left(\frac{k}{c} - l + 1\right)}.$$

But the series expansion of e^{-lT} is given by

$$e^{-lT} = \sum_{m=0}^{\infty} \frac{(-1)^m\, l^m\, T^m}{m!}.$$

Thus

$$\left[1 - \left(1 - \left(1 - e^{-T}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right]^i = \sum_{k,\,l\,m=0}^{\infty} \sum_{j=0}^{i} \frac{(-1)^{j+k+l+m}\, \Gamma\,(i+1)\, \Gamma\left(\frac{j}{d} + 1\right) \Gamma\left(\frac{k}{c} + 1\right) l^m T^m}{j!\,k!\,l!\,m!\,\Gamma\,(i - j + 1)\, \Gamma\left(\frac{j}{d} - k + 1\right) \Gamma\left(\frac{k}{c} - l + 1\right)}. \tag{16}$$

Substituting equation (16) into (15) and simplifying, we obtain

$$\mu_r^{'} = \sum_{i,\,k,\,l,\,m=0}^{\infty} \sum_{j=0}^{i} \frac{(-1)^{j+k+l+m}\, l^m\, \delta_{r,i}\, \Gamma\,(i+1)\, \Gamma\left(\frac{j}{d} + 1\right) \Gamma\left(\frac{k}{c} + 1\right)}{j!\,k!\,l!\,m!\,\Gamma\,(i - j + 1)\, \Gamma\left(\frac{j}{d} - k + 1\right) \Gamma\left(\frac{k}{c} - l + 1\right)} E\left(T^m\right).$$

□

Corollary 2. *Based on proposition* 1, *the* r^{th} *moment of the EGHL family is given by*

$$\mu_r^{'} = \sum_{i,\,k,\,l,\,m=0}^{\infty} \sum_{j=0}^{i} \frac{(-1)^{j+k+l+m}\, l^m\, \delta_{r,i}\, \Gamma\,(i+1)\, \Gamma\left(\frac{j}{d} + 1\right) \Gamma\left(\frac{k}{c} + 1\right)}{j!\,k!\,l!\,m!\,\Gamma\,(i - j + 1)\, \Gamma\left(\frac{j}{d} - k + 1\right) \Gamma\left(\frac{k}{c} - l + 1\right)} \left\{2 \sum_{n=0}^{\infty} \frac{(-1)^n\, \Gamma\,(m+1)}{\lambda^m\,(n+1)^m}\right\},$$

where $m = 1, 2, \ldots$.

Proposition 2. *The MGF of the EG T-X family of distributions is given by*

$$M_X(z) = \sum_{r,\, i,\, k,\, l,\, m=0}^{\infty} \sum_{j=0}^{i} \frac{(-1)^{j+k+l+m}\, z^r\, l^m\, \delta_{r,i}\, \Gamma\,(i+1)\, \Gamma\left(\frac{j}{d}+1\right) \Gamma\left(\frac{k}{c}+1\right)}{r!\, j!\, k!\, l!\, m!\, \Gamma\,(i-j+1)\, \Gamma\left(\frac{j}{d}-k+1\right) \Gamma\left(\frac{k}{c}-l+1\right)} E\left(T^m\right). \tag{17}$$

Proof. By definition the MGF is given by

$$M_X(z) = E\left(e^{zX}\right).$$

Using the series expansion of e^{zX}, gives us

$$M_X(z) = \sum_{r=0}^{\infty} \frac{z^r \mu_r^{'}}{r!}. \tag{18}$$

Substituting $\mu_r^{'}$ into equation (18), we have

$$M_X(z) = \sum_{r,\, i,\, k,\, l,\, m=0}^{\infty} \sum_{j=0}^{i} \frac{(-1)^{j+k+l+m}\, z^r\, l^m\, \delta_{r,i}\, \Gamma\,(i+1)\, \Gamma\left(\frac{j}{d}+1\right) \Gamma\left(\frac{k}{c}+1\right)}{r!\, j!\, k!\, l!\, m!\, \Gamma\,(i-j+1)\, \Gamma\left(\frac{j}{d}-k+1\right) \Gamma\left(\frac{k}{c}-l+1\right)} E\left(T^m\right),$$

which is the MGF. □

Corollary 3. *Based on proposition* 2*, the MGF of the EGHL family is*

$$M_X(z) = \sum_{r,\, i,\, k,\, l,\, m=0}^{\infty} \sum_{j=0}^{i} \frac{(-1)^{j+k+l+m}\, z^r\, l^m\, \delta_{r,i}\, \Gamma\,(i+1)\, \Gamma\left(\frac{j}{d}+1\right) \Gamma\left(\frac{k}{c}+1\right)}{r!\, j!\, k!\, l!\, m!\, \Gamma\,(i-j+1)\, \Gamma\left(\frac{j}{d}-k+1\right) \Gamma\left(\frac{k}{c}-l+1\right)} \times \left\{2 \sum_{n=0}^{\infty} \frac{(-1)^n\, \Gamma\,(m+1)}{\lambda^m\, (n+1)^m}\right\}.$$

Entropy is a measure of variation of uncertainty of a random variable. Entropy has been used extensively in several fields such as engineering and information theory. According to [3], the entropy of a random variable X with PDF $\mathrm{g}(x)$ is given by $\eta_X = -E\left\{\log\left(\mathrm{g}(X)\right)\right\}$.

Proposition 3. *The Shannon's entropy for the EG T-X family of distributions is given by*

$$\eta_X = -\log(cd) - \mu_T + \eta_T - E\left\{\log f\left[F^{-1}\left(1-\left(1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right)\right]\right\} + \left(\frac{1-d}{d}\right) E\left[\log\left(1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right)\right] + \left(\frac{1-c}{c}\right) E\left[\log\left(1-e^{-T}\right)\right], \quad (19)$$

where μ_T and η_T are the mean and the Shannon entropy of the random variable T.

Proof. By definition

$$\eta_X = (1-c)E\left\{\log\left[1-(1-F(X))^d\right]\right\} + (d-1)E\left[\log(1-F(X))\right] - E\left[\log f(X)\right] - \log(cd) + E\left\{\log\left[1-\left(1-(1-F(X))^d\right)^c\right]\right\} - E\left\{\log\left[r\left(-\log\left(1-(1-\bar{F}^d(x))^c\right)\right)\right]\right\}. \quad (20)$$

From Lemma 1, we know that $T = -\log[1-(1-\bar{F}^d(X))^c]$ and $X = F^{-1}\left\{1-\left[1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}$. Hence, we have

$$E\left[\log f(X)\right] = E\left\{\log f\left[F^{-1}\left(1-\left[1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right)\right]\right\}, \quad (21)$$

$$E\left[\log(1-F(X))\right] = E\left[\log\left(1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right], \quad (22)$$

$$E\left\{\log\left[1-(1-F(X))^d\right]\right\} = E\left[\log\left(1-e^{-T}\right)^{\frac{1}{c}}\right], \quad (23)$$

$$E\left\{\log\left[1-\left(1-(1-F(X))^{\frac{1}{d}}\right)^{\frac{1}{c}}\right]\right\} = E\left(-T\right), \quad (24)$$

and

$$E\left\{\log\left[r\left(-\log\left(1-(1-\bar{F}^d(x))^c\right)\right)\right]\right\} = E\left[\log r\left(T\right)\right]. \quad (25)$$

Substituting (21) through (25) into (20) yields

$$\eta_X = -\log(cd) - \mu_T + \eta_T - E\left\{\log f\left[F^{-1}\left(1-\left(1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right)\right]\right\} + \left(\tfrac{1-d}{d}\right) E\left[\log\left(1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right)\right] + \left(\tfrac{1-c}{c}\right) E\left[\log\left(1-e^{-T}\right)\right].$$

□

Substituting the mean and Shannon entropy of the half logistic distribution into (19), gives the Shannon entropy of the EGHL family.

Corollary 4. *From proposition* 3*, the Shannon entropy of the EGHL family is*

$$\eta_X = 2 - \log(2cd\lambda) - \frac{2\log(2)}{\lambda} - E\left\{\log f\left[F^{-1}\left(1-\left(1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right)\right]\right\} + \left(\frac{1-d}{d}\right)E\left[\log\left(1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right)\right] + \left(\frac{1-c}{c}\right)E\left[\log\left(1-e^{-T}\right)\right].$$

The mean of the half logistic distribution is $\mu_T = \frac{2\log(2)}{\lambda}$ *and the Shannon entropy is* $\eta_T = 2 - \log(2\lambda)$.

5 Parameter Estimation of Exponentiated Generalized Half Logistic Frétchet Distribution

Here, the estimation of the parameters of the EGHLFD was done using maximum likelihood estimation. Let $z_i = e^{-\left(\frac{a}{x_i}\right)^b}$ and $\bar{z}_i = 1 - e^{-\left(\frac{a}{x_i}\right)^b}$. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from EGHLFD, then the log-likelihood function for the vector of parameters $\boldsymbol{\vartheta} = (\lambda, c, d, b, a)^{'}$ is given by

$$\ell = n\log\left(2a^b bcd\lambda\right) + (d-1)\sum_{i=1}^{n}\log(\bar{z}_i) + (c-1)\sum_{i=1}^{n}\log(1-\bar{z}_i^d) + (\lambda-1)\sum_{i=1}^{n}\log\left[1-(1-\bar{z}_i^d)^c\right] - 2\sum_{i=1}^{n}\log\left\{1+\left[1-(1-\bar{z}_i^d)^c\right]^\lambda\right\} - (b-1)\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\left(\frac{a}{x_i}\right)^b. \quad (26)$$

Differentiating equation (26) with respect to the parameters λ, c, d, b and a, respectively and equating to zero gives

$$\frac{n}{\lambda} + \sum_{i=1}^{n}\log\left[1-(1-\bar{z}_i^d)^c\right] - 2\sum_{i=1}^{n}\frac{\left[1-(1-\bar{z}_i^d)^c\right]^\lambda\log\left[1-(1-\bar{z}_i^d)^c\right]}{1+\left[1-(1-\bar{z}_i^d)^c\right]^\lambda} = 0, \quad (27)$$

$$\frac{n}{c}+\sum_{i=1}^{n}\log(1-\bar{z}_i^{d})-(\lambda-1)\sum_{i=1}^{n}\frac{(1-\bar{z}_i^{d})^{c}\log(1-\bar{z}_i^{d})}{1-(1-\bar{z}_i^{d})^{c}}+ 2\sum_{i=1}^{n}\frac{\lambda(1-\bar{z}_i^{d})^{c}\left[1-(1-\bar{z}_i^{d})^{c}\right]^{\lambda-1}\log(1-\bar{z}_i^{d})}{1+\left[1-(1-\bar{z}_i^{d})^{c}\right]^{\lambda}}=0, \tag{28}$$

$$\frac{n}{d}+\sum_{i=1}^{n}\log(\bar{z}_i)-(c-1)\sum_{i=1}^{n}\frac{\bar{z}_i^{d}\log(\bar{z}_i)}{1-\bar{z}_i^{d}}+(\lambda-1)\sum_{i=1}^{n}\frac{c\bar{z}_i^{d}(1-\bar{z}_i^{d})^{c-1}\log(\bar{z}_i)}{1-(1-\bar{z}_i^{d})^{c}}- 2\sum_{i=1}^{n}\frac{\lambda c\bar{z}_i^{d}(1-\bar{z}_i^{d})^{c-1}\left[1-(1-\bar{z}_i^{d})^{c}\right]^{\lambda-1}\log(\bar{z}_i)}{1+\left[1-(1-\bar{z}_i^{d})^{c}\right]^{\lambda}}=0, \tag{29}$$

$$\frac{a^{-b}n(2a^{b}cd\lambda+2a^{b}bcd\lambda\log(a))}{2bcd\lambda}-\sum_{i=1}^{n}\log(x_i)-\sum_{i=1}^{n}\left(\frac{a}{x_i}\right)^{b}\log\left(\frac{a}{x_i}\right)+ (d-1)\sum_{i=1}^{n}\frac{z_i\left(\frac{a}{x_i}\right)^{b}\log\left(\frac{a}{x_i}\right)}{\bar{z}_i}-(c-1)\sum_{i=1}^{n}\frac{dz_i\bar{z}_i^{d-1}\left(\frac{a}{x_i}\right)^{b}\log\left(\frac{a}{x_i}\right)}{1-\bar{z}_i^{d}}+ (\lambda-1)\sum_{i=1}^{n}\frac{cdz_i\bar{z}_i^{d-1}(1-\bar{z}_i^{d})^{c-1}\left(\frac{a}{x_i}\right)^{b}\log\left(\frac{a}{x_i}\right)}{1-(1-\bar{z}_i^{d})^{c}}- 2\sum_{i=1}^{n}\frac{cd\lambda z_i\bar{z}_i^{d-1}(1-\bar{z}_i^{d})^{c-1}\left[1-(1-\bar{z}_i^{d})^{c}\right]^{\lambda-1}\left(\frac{a}{x_i}\right)^{b}\log\left(\frac{a}{x_i}\right)}{1+\left[1-(1-\bar{z}_i^{d})^{c}\right]^{\lambda}}=0, \tag{30}$$

$$\frac{bn}{a}-\sum_{i=1}^{n}\frac{b\left(\frac{a}{x_i}\right)^{b-1}}{x_i}+(d-1)\sum_{i=1}^{n}\frac{bz_i\left(\frac{a}{x_i}\right)^{b-1}}{x_i\bar{z}_i}-(c-1)\sum_{i=1}^{n}\frac{bdz_i\bar{z}_i^{d-1}\left(\frac{a}{x_i}\right)^{b-1}}{x_i(1-\bar{z}_i^{d})} +(\lambda-1)\sum_{i=1}^{n}\frac{bcdz_i\bar{z}_i^{d-1}(1-\bar{z}_i^{d})^{c-1}\left(\frac{a}{x_i}\right)^{b-1}}{x_i\left[1-(1-\bar{z}_i^{d})^{c}\right]}- 2\sum_{i=1}^{n}\frac{bcd\lambda z_i\bar{z}_i^{d-1}(1-\bar{z}_i^{d})^{c-1}\left[1-(1-\bar{z}_i^{d})^{c}\right]^{\lambda-1}\left(\frac{a}{x_i}\right)^{b-1}}{x_i\left\{1+\left[1-(1-\bar{z}_i^{d})^{c}\right]^{\lambda}\right\}}=0. \tag{31}$$

The maximum likelihood estimates of $\boldsymbol{\vartheta}=(\lambda,\, c,\, d,\, b,\, a)^{'}$ say $\hat{\boldsymbol{\vartheta}}=\left(\hat{\lambda},\, \hat{c},\, \hat{d},\, \hat{b},\, \hat{a}\right)^{'}$, are obtained by solving the non-linear equations (27), (28), (29), (30) and (31) using numerical methods.

6 Application

In this section, the application of the EGHLFD distribution is demonstrated using uncensored data on 100 observation on breaking stress of carbon fibers (in Gba) obtained from [6]. The data are: 0.39, 0.81, 0.85, 0.98, 1.08, 1.12, 1.17, 1.18, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.57, 1.59, 1.59, 1.61, 1.61, 1.69, 1.69, 1.71, 1.73, 1.80, 1.84, 1.84, 1.87, 1.89, 1.92, 2.00, 2.03, 2.03, 2.05, 2.12, 2.17, 2.17, 2.17, 2.35, 2.38, 2.41, 2.43, 2.48, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.76, 2.77, 2.79, 2.81, 2.81, 2.82, 2.83, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.51, 3.56, 3.60, 3.65, 3.68, 3.68, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90, 4.91, 5.08, 5.56. The fit of the EGHLFD was compared to that of transmuted Marshall-Olkin Frétchet distribution (TMOFD) and Marshall-Olkin Frétchet distribution (MOFD) using different goodness-of-fit tests including the Akaike information criterion (AIC), corrected Akaike information criterion (AICc), Bayesian information criterion (BIC), maximized log-likelihood under the model ($-2\hat{\ell}$), Anderson-Darling (A^*) and Cramér-Von Mises (W^*) statistics. The PDF of the TMOFD and MOFD are given by

$$\mathrm{g}(x) = \frac{\alpha b a^b x^{-(b+1)} e^{-\left(\frac{a}{x}\right)^b}}{\left[\alpha + (1-\alpha)e^{-\left(\frac{a}{x}\right)^b}\right]^2} \left[1 + \lambda - \frac{2\lambda e^{-\left(\frac{a}{x}\right)^b}}{\alpha + (1-\alpha)e^{-\left(\frac{a}{x}\right)^b}}\right], a > 0,\ b > 0,\ \alpha > 0,$$
$$|\lambda| \leq 1,\ x > 0,$$

and

$$\mathrm{g}(x) = b\left(\frac{\alpha}{a}\right)\left(\frac{a}{x}\right)^{b+1} e^{-\left(\frac{a}{x}\right)^b} \left[\alpha + (1-\alpha)e^{-\left(\frac{a}{x}\right)^b}\right]^{-2}, a > 0,\ b > 0,\ \alpha > 0,\ x > 0,$$

respectively. The maximum likelihood estimates (MLEs) of the parameters of the fitted distributions with their corresponding standard errors in bracket are given in Table 2.

From Table 3, it was clear that the EGHLFD provides a better fit to the carbon fibers data compared to other fitted distributions since it has the smallest value for all the goodness-of-fit statistics.

Table 2: MLEs of the parameters with their standard errors

Model	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{d}$	$\hat{\lambda}$
EGHLFD		23.986	0.267	6.814	4.384	73.770
		(4.989)	(0.283)	(17.638)	(5.556)	(1.290)
TMOFD	101.923	0.650	3.304			0.294
	(47.625)	(0.068)	(0.206)			(0.270)
MOFD	0.599	2.307	1.580			
	(0.309)	(0.489)	(0.160)			

Table 3: Goodness-of-fit Statistics

Model	$-2\hat{\ell}$	AIC	AICc	BIC	W*	A*
EGHLFD	**283.375**	**293.375**	**294.278**	**306.401**	**0.096**	**0.499**
TMOFD	301.973	309.973	310.611	320.393	0.238	1.268
MOFD	345.328	351.328	351.749	359.143	0.593	3.383

The plot of the empirical density and the density of the fitted models are shown in Figure 1.

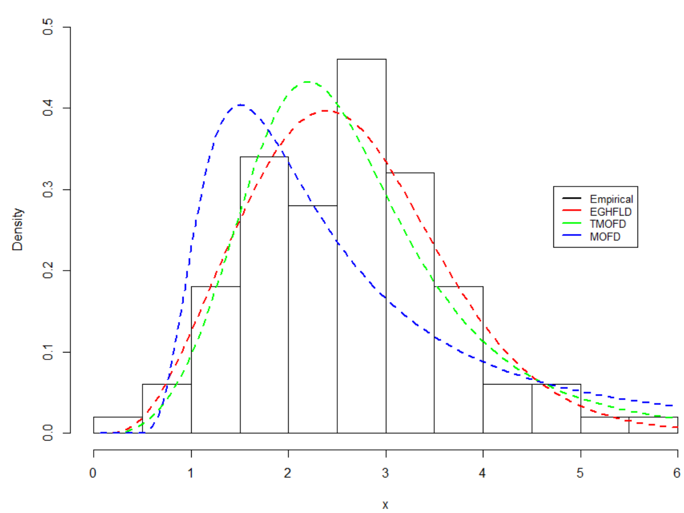

Figure 1: Plot of empirical density and density of fitted models

7 Conclusion

This study proposes the EG T-X family which is an extension of the T-X family of [1] and the exponentiated T-X family of [2] distributions. The new family has several sub-families as shown in Figure 2. The two extra shape parameters c and d provides greater flexibility for controlling skewness, kurtosis and possibly adding entropy to the center of the EG T-X density function. Specific example of a member of the EG T-X family of distribution, namely EGHLFD was given and its relationship with other baseline distributions established. Some statistical properties of the new family such as the quantile, moment, moment generating function, and Shannon entropy were derived.

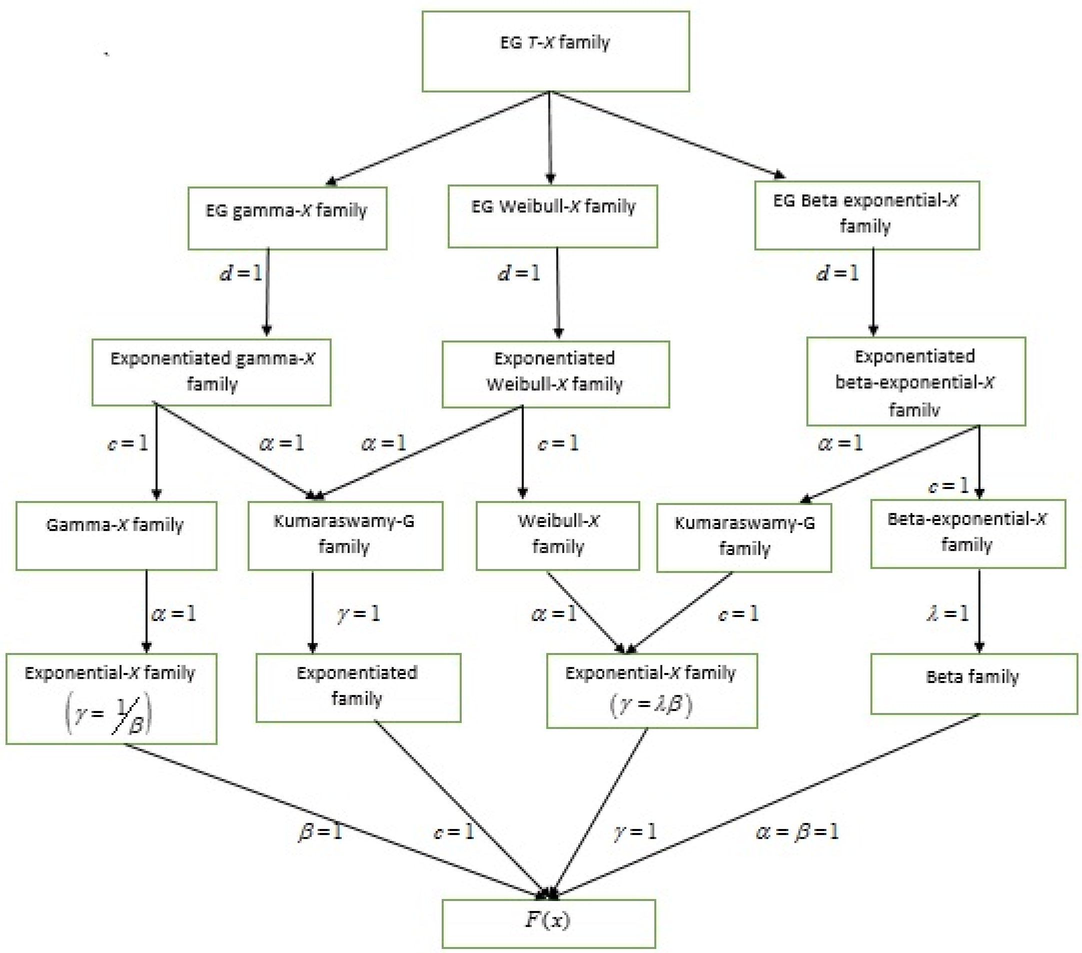

Figure 2: Families of EG T-X distributions

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References

[1] A. Alzaatreh, C. Lee and F. Famoye, A new method for generating families of continuous distributions, *Metron*, **71**(1), (2013), 63-79.

[2] A. Alzaghal, F. Famoye and C. Lee, Exponentiated T-X family of distributions with some applications, *International Journal of Statistics and Probability*, **2**(3), (2013), 31-49.

[3] C.E. Shannon, A mathematical theory of communication, *Bell System Technical Journal*, **27**, (2013), 379-432.

[4] G.M. Cordeiro, E.M.M. Ortega and C.C.D. da Cunha, The exponentiated generalized class of distributions, *Journal of Data Science*, **11**(1), (2013), 1-27.

[5] I.S. Gradshteyn and I.M. Ryzhik, *Tables of integrals, series and products*, Academic Press, New York, 2007.

[6] M.D. Nicholas and W. J. Padgett, A bootstrap control chart for Weibull percentiles, *Quality and Reliability Engineering International*, **22**, (2006), 141-151.

[7] N. Eugene, C. Lee and F. Famoye, The beta-normal distribution and its applications, *Communication in Statistics-Theory and Methods*, **31**(4), (2002), 497-512.