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Discussion on Generalized Modified Inverse Rayleigh

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Abstract: In this paper, a generalization of the modified inverse Rayleigh distribution called the new exponentiated generalized modified inverse Rayleigh distribution is proposed and studied. Various sub-models of the new distribution were discussed and statistical properties such as the quantile function, moment, moment generating function, Rényi entropy, reliability measure and order statistics were derived. The parameters of the new model were estimated using the method of maximum likelihood estimation and simulations were performed to assess the stability of the parameters with regards to the estimation method.

Keywords: Modfied inverse Rayleigh, quantile, moment, entropy, reliability measure.

1 Introduction

The selection of an appropriate distribution for modeling data sets plays an essential role in statistical analysis and forms the foundation to several parametric inferences. However, most of the data sets arising from different fields may not necessarily follow the existing distributions. Thus, researchers in the area of distribution theory have proposed several modifications of the existing distributions to enhance their performance in modeling data sets. Some of these modified distributions in literature include: generalized Weibull-exponential distribution [20], transmuted Erlang-truncated exponential distribution [18], odd generalized exponential generalized linear exponential distribution [12], weighted Weibull distribution [14], generalized Erlang-truncated exponential distribution [16], serial Weibull-Rayleigh distribution [15], McDonald exponentiated gamma distribution [1], Kumaraswamy transmuted modified Weibull distribution [13] and Kumaraswamy generalized power Weibull distribution [21]. These generalized models have the potential of modeling data sets with moderate and heavy tails, monotonic and non-monotonic failure rates. In addition, the generalization of the existing models tends to improve the flexibility and goodness-of-fit of the distributions against the intuition of model parsimony in many cases.
Recently, [7] proposed the modified inverse Rayleigh (MIR) distribution and studied its theoretical properties. The cumulative distribution function (CDF) of this distribution is given by

$$F(x) = e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}, x > 0, \qquad (1)$$

where $\alpha > 0$ and $\theta > 0$ are scale parameters.
This newly proposed distribution is suitable for modeling reliability pattern for engineering system or any process that exhibits either increasing or decreasing failure rates due to the flexibility of its hazard function in handling such failure rates. The MIR distribution contains both the inverse Rayleigh (IR) distribution and the inverse exponential (IE) distribution as sub-models. The MIR distribution is a special case of the modified inverse Weibull (MIW) distribution proposed by [8] with the following CDF

$$F(x) = e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^\eta}\right)}, x > 0, \qquad (2)$$

where $\alpha > 0$, $\theta > 0$ are scale parameters and $\eta > 0$ is a shape parameter.
The two parameters of the MIR distribution are all scale parameters. However, to control skewness and kurtosis, to model data with heavy tails and non-monotonic failure rates there is need for a distribution to have shape parameters. In order to address these issues and increase the flexibility of the MIR distribution, [9] studied the transmuted MIR distribution by adding a transmuting

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parameter to the existing baseline distribution. However, several modifications of the MIR distribution are yet to be proposed in literature using the various methods for developing generalized classes of distributions. Some of these methods of inducting new parameters into existing distributions include; beta-generated method [5], the exponentiated generalized class of distribution approach [4], the transformed-transformer (*T-X*) method [2], and the exponentiated *T-X* method [3].
[17] defined the CDF of the exponentiated generalized exponential-*X* family of distribution as

$$G(x)=\int_0^{-\log\left[1-\left(1-\bar{F}^d(x)\right)^c\right]}\lambda e^{-\lambda t}dt=1-\left\{1-\left[1-(1-F(x))^d\right]^c\right\}^\lambda, \quad (3)$$

where the random variable $T\in[0,\infty)$ has an exponential distribution with probability density function (PDF) $\lambda e^{-\lambda t},\lambda>0,t>0$, $X\in\mathbb{R}$ is a random variable with CDF $F(x)$, $\bar{F}(x)=1-F(x)$, and $c>0,d>0$ are shape parameters.
With the motivation of developing new distributions with tractable CDF to facilitate simulation, modeling data with different failure rates, generating distributions with heavier tails and modeling data from many field of studies with ease, this study proposes and investigates the theoretical properties of a new distribution called the new exponentiated generalized MIR (NEGMIR) distribution.
The rest of the paper is organized as follows: in section 2, the CDF, PDF, survival function and hazard function of the new distribution were defined. In section 3, some sub-models of the new distribution were discussed. In section 4, statistical properties of the new model were presented. In section 5, the parameters of the new distribution were estimated using maximum likelihood estimation. In section 6, simulation was performed to examine the stability of the model parameters. In section 7, applications of the new model was demonstrated using real data set. Finally, the concluding remarks of the study was given in section 8.

2 New Model

Suppose the random variable X has the CDF $e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)},x>0,\alpha\geq0,\theta\geq0,(\alpha+\theta>0)$, then the CDF of the NEGMIR distribution is given by

$$G(x)=1-\left\{1-\left[1-\left(1-e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}\right)^d\right]^c\right\}^\lambda,x>0, \quad (4)$$

where $\alpha\geq0,\theta\geq0$ are scale parameters and $\lambda>0,c>0,d>0$ are shape parameters. For positive integers λ and c, the physical interpretation to the CDF of the NGMIR distribution is as follows: given the lifetime of series-parallel system with independent components having CDF $1-(1-e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)})^d$. Suppose that the system is made up of λ independent components series subsystems and each of the subsystems consists of c independent parallel components. Suppose that $X_{ij}\sim1-(1-e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)})^d$, for $1\leq i\leq c$ and $1\leq j\leq\lambda$, represents the lifetime of the i^{th} component in the j^{th} subsystem and X is the lifetime of the entire system. Then,

$$\begin{aligned}\mathbb{P}(X\leq x)&=1-[1-\mathbb{P}(X_{11}\leq x,\ldots\leq X_{1c})]^\lambda\\&=1-[1-\mathbb{P}^c(X_{11}\leq x)]^\lambda\\&=1-\left\{1-\left[1-\left(1-e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}\right)^d\right]^c\right\}^\lambda,x>0.\end{aligned}$$

By differentiating equation (4), the PDF of the NEGMIR distribution is given by

$$g(x)=\lambda cd\left(\frac{\alpha}{x^2}+\frac{\theta}{x^3}\right)e^{-\left(\frac{\alpha}{x}+\frac{2\theta}{x^2}\right)}\left(1-e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}\right)^{d-1}\left[1-\left(1-e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}\right)^d\right]^{c-1}\times\left\{1-\left[1-\left(1-e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}\right)^d\right]^c\right\}^{\lambda-1},x>0. \quad (5)$$

Lemma 1. The PDF of the NEGMIR distribution can be written in a mixture form as

$$g(x)=\lambda cd\left(\frac{\alpha}{x^2}+\frac{2\theta}{x^3}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\xi_{ijkm}x^{-2m}e^{-(k+1)\left(\frac{\alpha}{x}\right)},x>0, \quad (6)$$

where

$$\xi_{ijkm}=\frac{(-1)^{i+j+k+m}(k+1)^m\theta^m\Gamma(\lambda)\Gamma(c(i+1))\Gamma(d(j+1))}{i!j!k!m!\Gamma(\lambda-i)\Gamma(c(i+1)-j)\Gamma(d(j+1)-k)},\Gamma(a+1)=a!$$

Proof. For a real non-integer $\eta>0$, a series representation for $(1-z)^{\eta-1}$, for $|z|<1$ is

$$(1-z)^{\eta-1}=\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(\eta)}{i!\Gamma(\eta-i)}z^i. \quad (7)$$

Using the series expansion in equation (7) thrice and the fact that $0<1-e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}<1$, we have

$$g(x)=\lambda cd\left(\frac{\alpha}{x^2}+\frac{2\theta}{x^3}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}\Gamma(\lambda)\Gamma(c(i+1))\Gamma(d(j+1))}{i!j!k!\Gamma(\lambda-i)\Gamma(c(i+1)-j)\Gamma(d(j+1)-k)}\times e^{-(k+1)\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}. \quad (8)$$

But

$$e^{-(k+1)\left(\frac{\theta}{x^2}\right)}=\sum_{m=0}^{\infty}\frac{(-1)^m(k+1)^m\theta^m x^{-2m}}{m!}. \quad (9)$$

Substituting equation (9) into equation (8), the mixture representation of the PDF of the NEGMIR distribution is obtained as

$$g(x)=\lambda cd\left(\frac{\alpha}{x^2}+\frac{2\theta}{x^3}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\xi_{ijkm}x^{-2m}e^{-(k+1)\left(\frac{\alpha}{x}\right)},x>0.$$

Figure 1 displays the different shapes of the NEGMIR distribution PDF. The PDF of NEGMIR distribution can

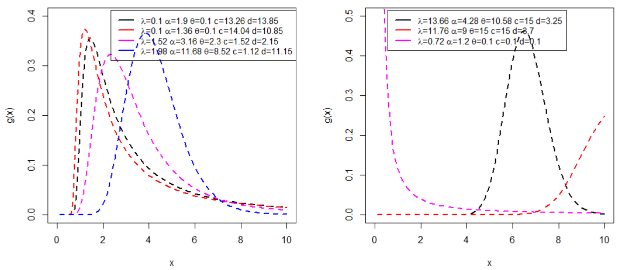

Fig. 1: NEGMIR distribution density function

be symmetric, left skewed, right skewed, J-shape, reversed J-shape or unimodal with small and large values of skewness and kurtosis for different parameter values.
The survival function of the NEGMIR distribution is

$$S(x)=\left\{1-\left[1-\left(1-e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}\right)^{d}\right]^{c}\right\}^{\lambda},x>0, \quad (10)$$

and the hazard function is given by

$$\tau(x)=\frac{\lambda cd\left(\frac{\alpha}{x^2}+\frac{2\theta}{x^3}\right)e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}\left(1-e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}\right)^{d-1}\left[1-\left(1-e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}\right)^{d}\right]^{c-1}}{1-\left[1-\left(1-e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}\right)^{d}\right]^{c}}, x>0. \quad (11)$$

The plots of hazard function shown in Figure 2 reveal different shapes such as monotonically decreasing, monotonically increasing, unimodal or upside down bathtub for different combination of the values of the parameters. These features make the NEGMIR distribution suitable for modeling different failure rates that are more likely to be encountered in real life situation.

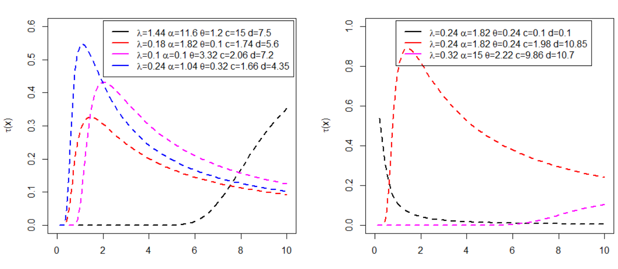

Fig. 2: Plots of the NEGMIR distribution hazard function

3 Sub-models

The NEGMIR distribution houses a number of sub-models that can be used in different fields for modeling data sets. These include: exponentiated generalized modified inverse Rayleigh (EGMIR) distribution, exponentiated generalized exponential inverse Rayleigh (EGEIR) distribution, exponentiated generalized inverse Rayleigh (EGIR) distribution, exponentiated generalized exponential inverse exponential (EGEIE) distribution, exponentiated generalized inverse exponential (EGIE) distribution, MIR distribution, IR distribution and IE distribution. A summary of the various sub-models of the NEGMIR distribution are given in Table 1

Table 1: Summary of sub-models from the NEGMIR distribution

Distribution	λ	α	θ	c	d
EGMIR	1	α	θ	c	d
EGEIR	λ	0	θ	c	d
EGIR	1	0	θ	c	d
EGEIE	λ	α	0	c	d
EGIE	1	α	0	c	d
MIR	1	α	θ	1	1
IR	1	0	θ	1	1
IE	1	α	0	1	1

4 Statistical Properties

In this section, the quantile, moment, moment generating function, reliability measure, entropy and order statistics were derived. Apart from the quantile function, all other statistical properties were derived using the parameter conditions $\alpha>0$, $\theta>0$, $\lambda>0$, $c>0$ and $d>0$.

4.1 Quantile Function

In order to simulate random numbers from the NEGMIR distribution, it is important to develop its quantile function.
Lemma 2. The quantile function of the NEGMIR distribution for $p\in(0,1)$ is

$$Q_X(p)=\begin{cases}\dfrac{2\theta}{-\alpha+\sqrt{\alpha^2-4\theta\log\left\{1-\left[1-\left(1-(1-p)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}}}, & \alpha>0,\ \theta>0,\\[2ex] \sqrt{\dfrac{\theta}{-\log\left\{1-\left[1-\left(1-(1-p)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}}}, & \alpha=0,\ \theta>0,\\[2ex] \dfrac{\alpha}{-\log\left\{1-\left[1-\left(1-(1-p)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}}, & \alpha>0,\ \theta=0.\end{cases} \quad (12)$$

For the case of $\alpha > 0$ and $\theta > 0$, the proof of the quantile is as follows.
Proof. By definition, the quantile function is given by

$$G(x_p) = \mathbb{P}(X \le x_p) = p.$$

Hence

$$\frac{\theta}{x_p^2} + \frac{\alpha}{x_p} + \log\left\{1 - \left[1 - \left(1 - (1-p)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\} = 0. \tag{13}$$

Letting $x_p = Q_X(p)$ in equation (13) and solving for $Q_X(p)$ gives

$$Q_X(p) = \frac{2\theta}{-\alpha + \sqrt{\alpha^2 - 4\theta \log\left\{1 - \left[1 - \left(1 - (1-p)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}}}.$$

For $p = 0.25, 0.5$ and 0.75, we get the first quartile, the median and the third quartile of the NGMIR distribution respectively.

4.2 Moment

Proposition 1.
The r^{th} non-central moment of the NEGMIR distribution is given by

$$\mu_r' = \lambda cd \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \xi_{ijkm}^{*}\left[\Gamma(2m-r+1) + \frac{2\theta}{\alpha^2(k+1)}\Gamma(2m-r+2)\right], \tag{14}$$

where r=1,2,... and

$$\xi_{ijkm}^{*} = \frac{(-1)^{i+j+k+m}(k+1)^{r-m-1}\theta^m \alpha^{r-2m}\Gamma(\lambda)\Gamma(c(i+1))\Gamma(d(j+1))}{i!\,j!\,k!\,m!\,\Gamma(\lambda-i)\Gamma(c(i+1)-j)\Gamma(d(j+1)-k)}.$$

Proof. By definition

$$\begin{aligned}
\mu_r' &= \int_0^\infty x^r g(x)dx \\
&= \int_0^\infty x^r \lambda cd\left(\frac{\alpha}{x^2} + \frac{2\theta}{x^3}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\xi_{ijkm}x^{-2m}e^{-(k+1)\left(\frac{\alpha}{x}\right)}dx \\
&= \lambda cd\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\xi_{ijkm}\int_0^\infty x^r\left(\frac{\alpha}{x^2} + \frac{2\theta}{x^3}\right)x^{-2m}e^{-(k+1)\left(\frac{\alpha}{x}\right)}dx \\
&= \lambda cd\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\xi_{ijkm}\int_0^\infty \left(\alpha x^{r-2m-2} + 2\theta x^{r-2m-3}\right)e^{-(k+1)\left(\frac{\alpha}{x}\right)}dx \\
&= \lambda cd\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\xi_{ijkm}\int_0^\infty \alpha x^{r-2m-2}e^{-(k+1)\left(\frac{\alpha}{x}\right)}dx \\
&+ \lambda cd\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\xi_{ijkm}\int_0^\infty 2\theta x^{r-2m-3}e^{-(k+1)\left(\frac{\alpha}{x}\right)}dx
\end{aligned}$$

Letting $y = \frac{\alpha(k+1)}{x}$ implies that if $x = 0, y = \infty$ and if $x = \infty, y = 0$. Also, $x = \frac{\alpha(k+1)}{y}$ and $dx = -\frac{x^2 dy}{\alpha(k+1)}$. Using the identity $\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt$,

$$\begin{aligned}
\mu_r' &= \lambda cd\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\xi_{ijkm}\times \\
&\left[\int_0^\infty \frac{1}{(k+1)}\left(\frac{\alpha(k+1)}{y}\right)^{r-2m}e^{-y}dy + \int_0^\infty \frac{2\theta}{\alpha(k+1)}\left(\frac{\alpha(k+1)}{y}\right)^{r-2m-1}e^{-y}dy\right] \\
&= \lambda cd\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\xi_{ijkm}\times \\
&\left[\alpha^{r-2m}(k+1)^{r-2m-1}\Gamma(2m-r+1) + 2\theta\alpha^{r-2m-2}(k+1)^{r-2m-2}\Gamma(2m-r+2)\right] \\
&= \lambda cd\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\xi_{ijkm}^{*}\left[\Gamma(2m-r+1) + \frac{2\theta}{\alpha^2(k+1)}\Gamma(2m-r+2)\right].
\end{aligned}$$

4.3 Moment Generating Function

Proposition 2. The moment generating function (MGF) of the NEGMIR distribution is

$$M_X(z) = \lambda cd\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\xi_{ijkm}^{**}\left[\Gamma(2m-r+1) + \frac{2\theta}{\alpha^2(k+1)}\Gamma(2m-r+2)\right], \tag{15}$$

where

$$\xi_{ijkm}^{**} = \frac{(-1)^{i+j+k+m}z^r(k+1)^{r-m-1}\theta^m\alpha^{r-2m}\Gamma(\lambda)\Gamma(c(i+1))\Gamma(d(j+1))}{i!\,j!\,k!\,m!\,r!\,\Gamma(\lambda-i)\Gamma(c(i+1)-j)\Gamma(d(j+1)-k)}.$$

Proof. By definition

$$\begin{aligned}
M_X(z) &= \int_0^\infty e^{zx}g(x)dx \\
&= \sum_{r=0}^{\infty}\frac{z^r}{r!}\int_0^\infty x^r g(x)dx \\
&= \lambda cd\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\xi_{ijkm}^{**}\left[\Gamma(2m-r+1) + \frac{2\theta}{\alpha^2(k+1)}\Gamma(2m-r+2)\right].
\end{aligned}$$

Note that the following series expansion $e^{zx} = \sum_{r=0}^{\infty}\frac{z^r x^r}{r!}$ was employed in the proof.

4.4 Entropy

In this subsection, the Rényi entropy of the random variable X is derived [19].
Proposition 3. The Rényi entropy of a random variable X having the NEGMIR distribution is

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[(\alpha\lambda cd)^{\delta}\sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\zeta_{ijkmn}\frac{\Gamma(2(\delta+m)+n-1)}{[\alpha(\delta+k)]^{2(\delta+m)+n-1}}\right], \tag{16}$$

where $\delta \neq 1$, $\delta > 0$ and

$$\zeta_{ijkmn} = \frac{(-1)^{i+j+k+m}\theta^m(\delta+k)^m\left(\frac{2\theta}{\alpha}\right)^m\Gamma(\delta+1)\Gamma(\delta(\lambda-1)+1)\Gamma(c(\delta+i)-\delta+1)\Gamma(d(\delta+j)-\delta+1)}{i!j!k!m!\Gamma(\delta-n+1)\Gamma(\delta(\lambda-1)-i+1)\Gamma(c(\delta+i)-\delta-j+1)\Gamma(d(\delta+j)-\delta-k+1)}.$$

Proof. The Rényi entropy [19] is defined as

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[\int_0^\infty g^{\delta}(x)dx\right], \delta \neq 1, \delta > 0.$$

Using the same method for expanding the density,

$$g^{\delta}(x) = (\alpha\lambda cd)^{\delta} \sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \zeta_{ijkmn} \left(\frac{1}{x}\right)^{2(\delta+m)+n} e^{-(\delta+k)\left(\frac{\alpha}{x}\right)}.$$

Hence

$$I_R(\delta) = \frac{1}{1-\delta} \times \log\left[(\alpha\lambda cd)^{\delta} \sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \zeta_{ijkmn} \int_0^{\infty} \left(\frac{1}{x}\right)^{2(\delta+m)+n} e^{-(\delta+k)\left(\frac{\alpha}{x}\right)} dx\right]$$

Letting $y = \frac{\alpha(\delta+k)}{x}$, when $x = 0, y = \infty$ and when $x = \infty, y = 0$. Also, $\frac{1}{x} = \frac{y}{\alpha(\delta+k)}$ and $dx = \frac{-x^2 dy}{\alpha(\delta+k)}$. Thus

$$\begin{aligned} I_R(\delta) &= \frac{1}{1-\delta} \log\left[(\alpha\lambda cd)^{\delta} \sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \zeta_{ijkmn} \int_0^{\infty} \frac{y^{2(\delta+m)+n-2}}{[\alpha(\delta+k)]^{2(\delta+m)+n-1}} e^{-y} dy\right] \\ &= \frac{1}{1-\delta} \log\left[(\alpha\lambda cd)^{\delta} \sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \zeta_{ijkmn} \frac{\Gamma(2(\delta+m)+n-1)}{[\alpha(\delta+k)]^{2(\delta+m)+n-1}}\right], \end{aligned}$$

where $\delta \neq 1$ and $\delta > 0$.

The Rényi entropy tends to Shannon entropy as $\delta \to 1$.

4.5 Reliability

Proposition 4. If X_1 is the strength of a component and X_2 is the stress, such that both follow the NEGMIR distribution with the same parameters, then the reliability is given by

$$R = 1 - \alpha\lambda cd \sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \varsigma_{ijkm} \left[\frac{\Gamma(2(m+1)-1)}{[\alpha(k+1)]^{2(m+1)-1}} + \frac{2\theta\,\Gamma(2(m+1))}{\alpha\,[\alpha(k+1)]^{2(m+1)}}\right], \tag{17}$$

where

$$\varsigma_{ijkm} = \frac{(-1)^{i+j+k+m}\,\theta^m\,(k+1)^m\,\Gamma(\lambda+1)\Gamma(c(i+1))\Gamma(d(j+1))}{i!\,j!\,k!\,m!\,\Gamma(\lambda-i+1)\Gamma(c(i+1)-j)\Gamma(d(j+1)-k)}.$$

Proof. By definition

$$\begin{aligned} R &= \mathbb{P}(X_2 < X_1) \\ &= \int_0^{\infty} g(x)G(x)dx \\ &= 1 - \int_0^{\infty} g(x)S(x)dx \\ &= 1 - \lambda cd \sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \varsigma_{ijkm} \int_0^{\infty} \alpha x^{-2(m+1)} e^{-(k+1)\left(\frac{\alpha}{x}\right)} dx \\ &\quad + 1 - \lambda cd \sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \varsigma_{ijkm} \int_0^{\infty} 2\theta x^{-(3+2m)} e^{-(k+1)\left(\frac{\alpha}{x}\right)} dx \end{aligned}$$

Letting $y = \frac{\alpha(k+1)}{x}$, when $x = 0, y = \infty$ and when $x = \infty, y = 0$. Also, $x = \frac{\alpha(k+1)}{y}$ and $dx = \frac{-x^2 dy}{\alpha(k+1)}$. Thus

$$\begin{aligned} R &= 1 - \lambda cd \sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \varsigma_{ijkm} \int_0^{\infty} \frac{\alpha y^{2(m+1)-2}}{[\alpha(k+1)]^{2(m+1)-1}} e^{-y} dy \\ &\quad + 1 - \lambda cd \sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \varsigma_{ijkm} \int_0^{\infty} \frac{2\theta y^{2m+1}}{[\alpha(k+1)]^{2(m+1)}} e^{-y} dy \\ &= 1 - \lambda cd \sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \varsigma_{ijkm} \left[\frac{\alpha\Gamma(2(m+1)-1)}{[\alpha(k+1)]^{2(m+1)-1}} + \frac{2\theta\Gamma(2(m+1))}{[\alpha(k+1)]^{2(m+1)}}\right] \\ &= 1 - \alpha\lambda cd \sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \varsigma_{ijkm} \left[\frac{\Gamma(2(m+1)-1)}{[\alpha(k+1)]^{2(m+1)-1}} + \frac{2\theta\,\Gamma(2(m+1))}{\alpha\,[\alpha(k+1)]^{2(m+1)}}\right]. \end{aligned}$$

4.6 Order Statistics

Order statistics have a very useful role in statistics and probability. Hence, in this section the PDF of p^{th} order statistic of the NEGMIR distribution was developed. Suppose $X_1, X_2, \ldots, X_n$ is a random sample having the NEGMIR distribution and $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ are order statistics obtained from the sample. The PDF, $g_{p:n}(x)$, of the p^{th} order statistic $X_{p:n}$ is

$$g_{p:n}(x) = \frac{1}{B(p, n-p+1)} [G(x)]^{p-1} [1 - G(x)]^{n-p} g(x),$$

where $G(x)$ and $g(x)$ are the CDF and PDF of the NEGMIR distribution respectively, and $B(\cdot,\cdot)$ is the beta function. Since $0 < G(x) < 1$ for $x > 0$, using the binomial series expansion of $[1 - G(x)]^{n-p}$, which is given by

$$[1 - G(x)]^{n-p} = \sum_{l=0}^{n-p} (-1)^l \binom{n-p}{l} [G(x)]^l,$$

we have

$$g_{p:n}(x) = \frac{1}{B(p,n-p+1)} \sum_{l=0}^{n-p} (-1)^l \binom{n-p}{l} [G(x)]^{p+l-1} g(x). \tag{18}$$

Substituting the CDF and PDF of the NEGMIR distribution into equation (18) gives

$$g_{p:n}(x) = \sum_{l=0}^{n-p}\sum_{m=0}^{p+l-1} \frac{(-1)^{l+m}\, n!\,(p+l-1)!}{l!\,(m+1)!\,(p-1)!\,(n-p-l)!\,(p+l-m-1)!} g(x;\alpha,\lambda_{m+1},\theta,c,d), \tag{19}$$

where $g(x;\alpha,\lambda_{m+1},\theta,c,d)$ is the PDF of the NEGMIR distribution with parameters α,θ,c,d and $\lambda_{m+1} = \lambda(m+1)$. It is clear to see that the density of the p^{th} order statistic given in equation (19) is a weighted function of the NEGMIR distribution with different shape parameters.

Proposition 5. The r^{th} non-central moment of the p^{th} order statistic is given by

$$\mu_r'^{(p:n)} = \lambda cd \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{n-p}\sum_{m=0}^{p+l-1} \varphi_{ijklmq} \left[\Gamma(2q-r+1) + \frac{2\theta\,\Gamma(2q-r+2)}{\alpha^2(k+1)}\right], \tag{20}$$

where $r = 1, 2, \ldots$ and

$$\varphi_{ijklmq} = \frac{(-1)^{i+j+k+l+m+q}(k+1)^{r-q-1}\theta^q\alpha^{r-2q}\Gamma(n+1)\Gamma(p+l)\Gamma(\lambda(m+1))\Gamma(c(i+1))\Gamma(d(j+1))}{i!j!k!l!m!q!(p-1)!(n-p-l)!\Gamma(p+l-m)\Gamma(\lambda(m+1)-i)\Gamma(c(i+1)-j)\Gamma(d(j+1)-k)}.$$

Proof. By definition

$$\begin{aligned} \mu_r'^{(p:n)} &= \int_0^{\infty} x^r g_{p:n}(x)dx \\ &= \int_0^{\infty} x^r \sum_{l=0}^{n-p}\sum_{m=0}^{p+l-1} \frac{(-1)^{l+m}\, n!\,(p+l-1)!}{l!\,(m+1)!\,(p-1)!\,(n-p-l)!\,(p+l-m-1)!} g(x;\alpha,\lambda_{m+1},\theta,c,d)dx \\ &= \sum_{l=0}^{n-p}\sum_{m=0}^{p+l-1} \frac{(-1)^{l+m}\, n!\,(p+l-1)!}{l!\,(m+1)!\,(p-1)!\,(n-p-l)!\,(p+l-m-1)!} \int_0^{\infty} x^r g(x;\alpha,\lambda_{m+1},\theta,c,d)dx. \end{aligned}$$

Employing the same method for deriving the non-central moment, we obtain

$$\mu_r'^{(p:n)} = \lambda cd \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{n-p}\sum_{m=0}^{p+l-1} \varphi_{ijklmq} \left[\Gamma(2q-r+1) + \frac{2\theta\,\Gamma(2q-r+2)}{\alpha^2(k+1)}\right].$$

5 Parameter Estimation

In this section, the estimation of the unknown parameter vector $\boldsymbol{\vartheta} = (\lambda, \alpha, \theta, c, d)'$ using the method of maximum likelihood estimation was carried out. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from NEGMIR distribution. Let $z_i = e^{-\left(\frac{\alpha}{x_i}+\frac{\theta}{x_i^2}\right)}$ and $\bar{z_i} = 1 - e^{-\left(\frac{\alpha}{x_i}+\frac{\theta}{x_i^2}\right)}$, then the log-likelihood function is given by

$$\ell = n\log(cd\lambda) + (d-1)\sum_{i=1}^{n}\log(\bar{z}_i) + (c-1)\sum_{i=1}^{n}\log(1-\bar{z}_i^d) + (\lambda-1)\sum_{i=1}^{n}\log\left[1-(1-\bar{z}_i^d)^c\right]$$
$$+\sum_{i=1}^{n}\log\left(\frac{\alpha}{x_i^2}+\frac{\theta}{x_i^3}\right) - \sum_{i=1}^{n}\left(\frac{\alpha}{x_i}+\frac{\theta}{x_i^2}\right). \tag{21}$$

By differentiating the log-likelihood function with respect to the parameters λ, c, d, α and θ, the score functions are obtained as

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\log\left[1-(1-\bar{z}_i^d)^c\right], \tag{22}$$

$$\frac{\partial\ell}{\partial c} = \frac{n}{c} + \sum_{i=1}^{n}\log\left(1-\bar{z}_i^d\right) - (\lambda-1)\sum_{i=1}^{n}\frac{(1-\bar{z}_i^d)^c\log(1-\bar{z}_i^d)}{1-(1-\bar{z}_i^d)^c}, \tag{23}$$

$$\frac{\partial\ell}{\partial d} = \frac{n}{d} + \sum_{i=1}^{n}\log(\bar{z}_i) - (c-1)\sum_{i=1}^{n}\frac{\bar{z}_i^d\log(\bar{z}_i)}{1-\bar{z}_i^d} + (\lambda-1)\sum_{i=1}^{n}\frac{c\bar{z}_i^d(1-\bar{z}_i^d)^{c-1}\log(\bar{z}_i)}{1-(1-\bar{z}_i^d)^c}, \tag{24}$$

$$\frac{\partial\ell}{\partial\alpha} = \sum_{i=1}^{n}\frac{1}{x_i^2\left(\frac{\alpha}{x_i^2}+\frac{\theta}{x_i^3}\right)} - \sum_{i=1}^{n}\frac{1}{x_i} + (d-1)\sum_{i=1}^{n}\frac{z_i}{x_i\bar{z}_i} - (c-1)\sum_{i=1}^{n}\frac{dz_i\bar{z}_i^{d-1}}{x_i(1-\bar{z}_i^d)} +$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{cdz_i\bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}}{x_i\left[1-(1-\bar{z}_i^d)^c\right]}, \tag{25}$$

$$\frac{\partial\ell}{\partial\theta} = \sum_{i=1}^{n}\frac{1}{x_i^3\left(\frac{\alpha}{x_i^2}+\frac{\theta}{x_i^3}\right)} - \sum_{i=1}^{n}\frac{1}{x_i^2} + (d-1)\sum_{i=1}^{n}\frac{z_i}{x_i^2\bar{z}_i} - (c-1)\sum_{i=1}^{n}\frac{dz_i\bar{z}_i^{d-1}}{x_i^2(1-\bar{z}_i^d)} +$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{cdz_i\bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]}. \tag{26}$$

Equating the score functions to zero and solving for the unknown parameters in the system of non-linear equations numerically yields the maximum likelihood estimates of the parameters. For the purpose of constructing confidence intervals for the parameters, the observed information matrix $J(\boldsymbol{\vartheta})$ is used due to the complex nature of the expected information matrix. The observed information matrix is given by

$$J(\boldsymbol{\vartheta}) = -\begin{bmatrix} \frac{\partial^2\ell}{\partial\lambda^2} & \frac{\partial^2\ell}{\partial\lambda\partial c} & \frac{\partial^2\ell}{\partial\lambda\partial d} & \frac{\partial^2\ell}{\partial\lambda\partial\alpha} & \frac{\partial^2\ell}{\partial\lambda\partial\theta} \\ & \frac{\partial^2\ell}{\partial c^2} & \frac{\partial^2\ell}{\partial c\partial d} & \frac{\partial^2\ell}{\partial c\partial\alpha} & \frac{\partial^2\ell}{\partial c\partial\theta} \\ & & \frac{\partial^2\ell}{\partial d^2} & \frac{\partial^2\ell}{\partial d\partial\alpha} & \frac{\partial^2\ell}{\partial d\partial\theta} \\ & & & \frac{\partial^2\ell}{\partial\alpha^2} & \frac{\partial^2\ell}{\partial\alpha\partial\theta} \\ & & & & \frac{\partial^2\ell}{\partial\theta^2} \end{bmatrix}.$$

The elements of the observed information matrix are given in the appendix. When the usual regularity conditions are satisfied and that the parameters are within the interior of the parameter space, but not on the boundary, the distribution of $\sqrt{n}(\hat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta})$ converges to the multivariate normal distribution $N_5(\mathbf{0}, I^{-1}(\boldsymbol{\vartheta}))$, where $I(\boldsymbol{\vartheta})$ is the expected information matrix. The asymptotic behavior remains valid when $I(\boldsymbol{\vartheta})$ is replaced by the observed information matrix evaluated at $J(\hat{\boldsymbol{\vartheta}})$. The asymptotic multivariate normal distribution $N_5(\mathbf{0}, J^{-1}(\hat{\boldsymbol{\vartheta}}))$ is a very useful tool for constructing an approximate $100(1-\psi)\%$ two-sided confidence intervals for the model parameters, where ψ is the significance level.

6 Monte carlo Simulation

In this section, the properties of the maximum likelihood estimators of the parameters of the NEGMIR distribution were examined using simulation. The average bias (AB), the root mean square error (RMSE) and the average width (AW) of the parameter values were observed. The quantile function given in equation (12) was used to generate random samples from the NEGMIR distribution. The simulation experiment was repeated for $N = 1,000$ times each with sample sizes $n = 25, 50, 75, 100, 200, 300, 600$ and parameter values $I: \lambda = 0.5, \alpha = 0.1, \theta = 0.8, c = 0.4, d = 0.5)$ and $II: \lambda = 0.4, \alpha = 0.5, \theta = 0.5, c = 2.5, d = 1.5$. From Table 2, both the AB and the RMSE of the parameters decreases to zero as the sample size increases. Also, the AW for the confidence intervals of the parameters decreases as the sample size increases. Thus, the maximum likelihood estimates and their asymptotic properties can be employed for estimating and constructing confidence intervals even for reasonably small sample size.

7 Applications

In this section, applications of the NEGMIR distribution were demonstrated using two real data sets. The goodness-of-fit of the NEGMIR distribution was compared with that of its sub-models and the new generalized inverse Weibull (NGIW) distribution using Kolmogorov-Smirnov (K-S) statistic and Cramér-von Misses distance (W^*) values, as well as Akaike information criterion (AIC), corrected Akaike information criterion (AICc) and Bayesian information criterion (BIC). The PDF of the NGIW distribution is given by

$$g(x) = \beta\left(\alpha + \eta\theta\left(\frac{1}{x}\right)^{\eta-1}\right)\left(\frac{1}{x}\right)^2 e^{\left(-\frac{\alpha}{x}-\theta\left(\frac{1}{x}\right)^{\eta}\right)}\left(1-e^{\left(-\frac{\alpha}{x}-\theta\left(\frac{1}{x}\right)^{\eta}\right)}\right)^{\beta-1}, x > 0, \tag{27}$$

where $\eta > 0, \beta > 0$ are the shape parameters and $\alpha > 0$, $\theta > 0$ are scale parameters of the distribution.

Table 2: Monte Carlo simulation results: AB, RMSE and AW

		I			II		
Parameter	n	AB	RMSE	AW	AB	RMSE	AW
λ	25	0.533	0.975	17.759	0.430	2.404	39.319
	50	0.448	0.827	14.581	0.222	1.298	20.377
	75	0.392	0.756	12.071	0.251	1.089	19.377
	100	0.377	0.702	10.353	0.159	0.625	12.140
	200	0.383	0.692	9.412	0.167	0.607	11.807
	300	0.333	0.573	7.749	0.190	0.607	12.990
	600	0.267	0.501	5.482	0.183	0.507	9.329
α	25	0.389	0.523	6.935	-0.103	0.454	26.910
	50	0.341	0.476	5.236	-0.109	0.409	15.649
	75	0.301	0.444	4.524	-0.092	0.384	14.265
	100	0.292	0.427	4.047	-0.113	0.365	11.114
	200	0.239	0.392	2.762	-0.076	0.321	11.240
	300	0.223	0.361	2.470	-0.043	0.297	13.128
	600	0.157	0.307	1.539	-0.020	0.247	11.125
θ	25	0.191	0.701	10.047	-0.018	0.419	37.024
	50	0.032	0.433	7.782	0.050	0.380	25.619
	75	-0.009	0.378	7.046	0.103	0.393	22.394
	100	-0.035	0.334	6.382	0.132	0.382	20.964
	200	-0.079	0.287	4.757	0.128	0.352	21.270
	300	-0.082	0.272	4.697	0.204	0.367	22.982
	600	-0.073	0.247	4.098	0.026	0.351	18.444
c	25	0.050	0.222	6.062	3.747	9.793	508.001
	50	0.059	0.184	4.900	1.801	4.460	241.696
	75	0.064	0.161	4.507	1.021	3.243	204.629
	100	0.063	0.157	4.213	0.758	2.839	141.913
	200	0.081	0.167	3.170	0.131	1.794	98.608
	300	0.076	0.158	3.094	-0.159	1.430	81.678
	600	0.069	0.149	2.584	-0.422	0.977	59.848
d	25	-0.123	0.010	5.551	0.089	0.028	29.222
	50	-0.120	0.008	4.413	0.046	0.024	21.361
	75	-0.107	0.008	4.070	-0.018	0.023	19.872
	100	-0.106	0.007	3.764	-0.016	0.023	20.184
	200	-0.123	0.007	3.171	-0.034	0.022	18.868
	300	-0.110	0.006	2.879	-0.023	0.021	19.484
	600	-0.091	0.006	2.344	-0.015	0.020	17.662

Table 3: Failure times data for the air conditioning system of an aircraft

23	261	87	7	120	14	62	47	225	71
246	21	42	20	5	12	120	11	3	14
71	11	14	11	16	90	1	16	52	95

7.1 Aircraft Data

The data comprises failure times for the air conditioning system of an aircraft from a random sample of 30 observations. The data set can be found in [11] and [10]. The data set is given in Table 3.

The maximum likelihood estimates of the parameters and their corresponding standard errors in bracket are displayed in Table 4. The parameters of the NEGMIR distribution were all significant at the 5% significance level.

The NEGMIR distribution provides a better to the data set than its sub-models and the NGIW distribution. From Table 5, the NEGMIR distribution has the highest log-likelihood and the smallest K-S, W*, AIC, AICc, and BIC values compared to the other fitted models. Although the NEGMIR distribution is the best model, the NGIW distribution also provides a good fit to the data set.

To make a complete inference about a model, it is necessary to reduce the number of parameters of the model and investigate its effect on the reduced model with regards to providing good fit to a data set. Thus, the likelihood ratio test (LRT) was therefore performed to compare the NEGMIR distribution with its sub-models. The LRT statistic and their corresponding *P*-values in Table 6 revealed that the NGMIR distribution provides a

Table 4: Maximum likelihood estimates of parameters and standard errors for aircraft data

Model	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{c}$	$\hat{d}$
NEGMIR	0.082	18.949	3.736	0.132	11.356
	(0.018)	(2.491)	(0.851)	(0.025)	(1.309)
EGMIR		29.072	1.569	0.326	0.674
		(12.559)	(0.566)	(0.133)	(0.153)
NEGIR	47.262		10.089	0.897	0.003
	(1.6×10^{-4})		(0.006)	(0.164)	(2.54×10^{-3})
NEGIE	0.062	1.734		13.278	6.537
	(0.016)	(0.234)		(14.581)	(1.014)
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\eta}$	
NGIW	7.312	0.628	0.944	150.959	
	(2.226)	(0.150)	(0.994)	(158.932)	

Table 5: Log-likelihood, goodness-of-fit statistics and information criteria for aircraft data

Model	ℓ	AIC	AICc	BIC	K-S	W*
NEGMIR	**-146.520**	**303.046**	**306.698**	**309.882**	**0.1490**	**0.0701**
EGMIR	-151.920	311.842	314.342	317.312	0.2336	0.1636
NEGIR	-158.360	324.723	327.223	330.192	0.3111	0.5359
NEGIE	-156.420	320.840	323.340	326.309	0.2816	0.5021
NGIW	-148.500	304.993	307.493	310.462	0.2270	0.1538

Table 6: Likelihood ratio test statistic for aircraft data

Model	Hypotheses	LRT	*P*-values
EGMIR	$H_0:\lambda=1$ vs $H_1:H_0$ is false	10.797	<0.001
NEGIR	$H_0:\alpha=0$ vs $H_1:H_0$ is false	23.677	<0.001
NEGIE	$H_0:\theta=0$ vs $H_1:H_0$ is false	19.794	<0.001

good fit than its sub-models.
The asymptotic variance-covariance matrix for the estimated parameters of the NEGMIR distribution is given by

$$J^{-1}=\begin{bmatrix} 3.191\times10^{-4} & 6.003\times10^{-5} & -8.433\times10^{-3} & 1.347\times10^{-2} & 8.209\times10^{-4} \\ 6.003\times10^{-5} & 6.306\times10^{-4} & 1.052\times10^{-2} & 3.744\times10^{-2} & 1.416\times10^{-2} \\ -8.433\times10^{-3} & 1.052\times10^{-2} & 1.714 & 0.624 & 0.488 \\ 1.347\times10^{-2} & 3.744\times10^{-2} & 0.624 & 6.205 & 1.236 \\ 8.209\times10^{-4} & 1.416\times10^{-2} & 0.488 & 1.236 & 0.724 \end{bmatrix}.$$

Hence, the approximate 95% confidence interval for the parameters λ,α,θ,c and d are $[0.0468, 0.1168]$, $[14.0665, 23.8311]$, $[2.0681, 5.4043]$, $[0.0827, 0.1811]$ and $[8.7900, 13.9218]$ respectively. Figure 3 displays the empirical density and the fitted densities of the distributions.

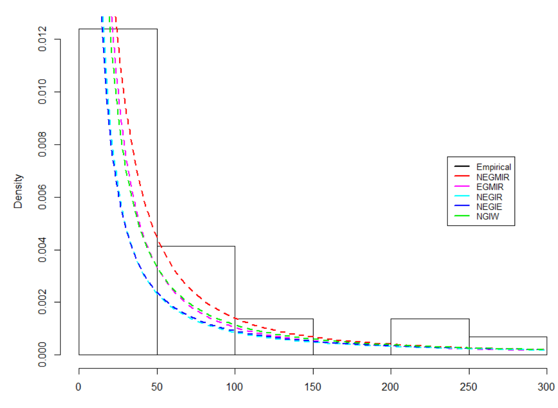

Fig. 3: Empirical and fitted densities plot for aircraft data

Table 7: March precipitation in Minneapolis/St Paul

0.77	1.74	0.81	1.20	1.95	1.20	0.47	1.43	3.37	2.20
3.00	3.09	1.51	2.10	0.52	1.62	1.31	0.32	0.59	0.81
2.81	1.87	1.18	1.35	4.75	2.48	0.96	1.89	0.90	2.05

Table 8: Maximum likelihood estimates of parameters and standard errors for precipitation data

Model	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{c}$	$\hat{d}$
NEGMIR	0.225	3.022	2.246	0.112	24.039
	(0.102)	(0.515)	(0.281)	(0.052)	(12.399)
EGMIR		1.658	2.918	0.235	1.877
		(0.138)	(0.355)	(0.051)	(0.146)
NEGIR	0.087		1.305	0.219	10.813
	(0.018)		(0.181)	(0.028)	(1.555)
NEGIE	8.228	9.708		0.258	0.092
	(4.261)	(2.387)		(0.086)	(0.022)
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\eta}$	
NGIW	2.202	3.292	4.635×10^{-5}	5.822	
	(0.448)	(1.087)	(0.002)	(0.014)	

7.2 Precipitation Data

The data was first reported by Hinkley [6] and consists of 30 observations of March precipitation (in inches) in Minneapolis/ St Paul. The data set is given in Table 7.
The maximum likelihood estimates for the parameters of the fitted distributions and their corresponding standard errors in brackets are shown in Table 8. The NEGMIR distribution had all its parameters to be significant at the 5% significance level except d which was significant at 10%.The parameters of the EGMIR, NEGIR and NEGIE distributions were all significant. The parameters of the NGIW distribution were also significant, except θ.
Table 9 revealed that the NEGMIR distribution provides a better fit to the precipitation data compared to its sub-models and the NGIW distribution since it has the highest log-likelihood, smallest K-S, W*, AIC, AICc and BIC values.
The LRT was performed to compare the NEGMIR distribution with its sub-models. The results as shown in Table 10 revealed the NEGMIR distribution provides a better fit to the precipitation data than its sub-models.
The estimated asymptotic variance-covariance matrix of the NEGMIR distribution for the precipitation data is given by

$$J^{-1}=\begin{bmatrix} 0.010 & 0.003 & -1.033 & 0.017 & 0.004 \\ 0.003 & 0.003 & -0.307 & -0.001 & -0.003 \\ -1.033 & -0.307 & 153.725 & 0.287 & 0.099 \\ 0.017 & -0.001 & 0.287 & 0.265 & 0.046 \\ 0.004 & -0.003 & 0.099 & 0.046 & 0.079 \end{bmatrix}.$$

Table 9: Log-likelihood, goodness-of-fit statistics and information criteria for precipitation data

Model	ℓ	AIC	AICc	BIC	K-S	W*
NEGMIR	**-37.870**	**85.738**	**89.390**	**92.744**	**0.076**	**0.014**
EGMIR	-42.750	93.492	96.101	99.097	0.208	0.138
NEGIR	-40.210	88.421	91.030	94.025	0.282	0.071
NEGIE	-40.460	88.912	91.521	94.517	0.140	0.070
NGIW	-39.66	87.326	89.935	92.931	0.125	0.066

Table 10: Likelihood ratio test statistic for precipitation data

Model	Hypotheses	LRT	P-values
EGMIR	$H_0:\lambda=1$ vs $H_1:H_0$ is false	9.754	0.002
NEGIR	$H_0:\alpha=0$ vs $H_1:H_0$ is false	4.682	0.030
NEGIE	$H_0:\theta=0$ vs $H_1:H_0$ is false	5.174	0.023

The approximate 95% confidence interval for the parameters λ,α,θ,c and d are $[0.025,0.424]$, $[2.012,4.032]$, $[1.696,2.797]$, $[0.011,0.214]$ and $[-0.262,48.340]$ respectively. Figure 4 displays the empirical density and the fitted densities of the distributions.

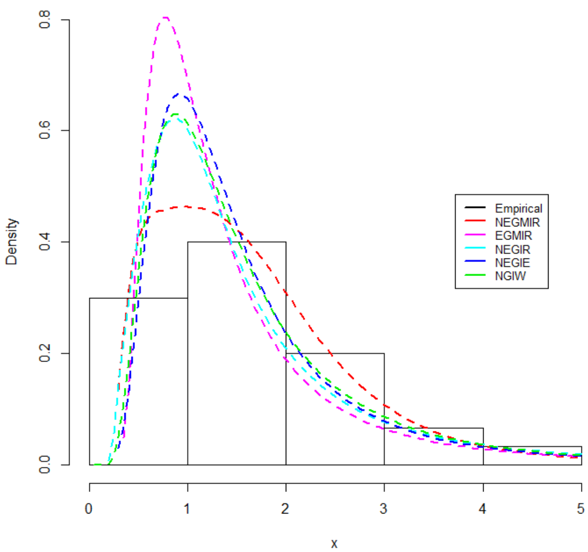

Fig. 4: Empirical and fitted densities plot for precipitation data

8 Conclusion

This study proposes a five-parameter distribution called NEGMIR distribution, which is an extension of the MIR distribution and contains several sub-models suitable for for modeling data from different fields of study. Various statistical properties of the new distribution such as the quantile function, moment, moment generating function, Rényi entropy, reliability measure and order statistics were studied. The parameters of the model were estimated using the method of maximum likelihood estimation and simulation studies performed to examine the estimators of the parameters. The usefulness of the new model was demonstrated using two real data sets.

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Appendix

The elements of the 5×5 unit observed information matrix are given by:

$$\frac{\partial^2\ell}{\partial\lambda^2}=-\frac{n}{\lambda^2},$$

$$\frac{\partial^2\ell}{\partial\lambda\partial c}=-\sum_{i=1}^{n}\frac{(1-\bar{z}_i^d)^c\log(1-\bar{z}_i^d)}{1-(1-\bar{z}_i^d)^c},$$

$$\frac{\partial^2\ell}{\partial\lambda\partial d}=\sum_{i=1}^{n}\frac{c\bar{z}_i^d(1-\bar{z}_i^d)^{c-1}\log(\bar{z}_i)}{1-(1-\bar{z}_i^d)^c},$$

$$\frac{\partial^2\ell}{\partial\lambda\partial\alpha}=\sum_{i=1}^{n}\frac{cdz_i\bar{z}_i^{d-1}(1-\bar{z}^d)^{c-1}}{x_i\left[1-(1-\bar{z}_i^d)^c\right]},$$

$$\frac{\partial^2\ell}{\partial\lambda\partial\theta}=\sum_{i=1}^{n}\frac{cdz_i\bar{z}_i^{d-1}(1-\bar{z}^d)^{c-1}}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]},$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial c^2}&=-\frac{n}{c^2}-(\lambda-1)\sum_{i=1}^{n}\frac{(1-\bar{z}_i^d)^{2c}\log(1-\bar{z}_i^d)^2}{\left[1-(1-\bar{z}_i^d)^c\right]^2}\\&\quad-(\lambda-1)\sum_{i=1}^{n}\frac{(1-\bar{z}_i^d)^c\log(1-\bar{z}_i^d)^2}{1-(1-\bar{z}_i^d)^c},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial c\partial d}&=-\sum_{i=1}^{n}\frac{\bar{z}_i^d\log(\bar{z}_i)}{1-\bar{z}_i^d}\\&\quad+(\lambda-1)\sum_{i=1}^{n}\frac{\bar{z}_i^d(1-\bar{z}_i^d)^{c-1}\log(\bar{z}_i)}{1-(1-\bar{z}_i^d)^c}\\&\quad+(\lambda-1)\sum_{i=1}^{n}\frac{c\bar{z}_i^d(1-\bar{z}_i^d)^{2c-1}\log(\bar{z}_i)\log(1-\bar{z}_i^d)}{\left[1-(1-\bar{z}_i^d)^c\right]^2}\\&\quad+(\lambda-1)\sum_{i=1}^{n}\frac{c\bar{z}_i^d(1-\bar{z}_i^d)^{c-1}\log(\bar{z}_i)\log(1-\bar{z}_i^d)}{1-(1-\bar{z}_i^d)^c},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial c\partial\alpha}&=(\lambda-1)\sum_{i=1}^{n}\frac{dz_i\bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}}{x_i\left[1-(1-\bar{z}_i^d)^c\right]}+(\lambda-1)\sum_{i=1}^{n}\frac{cdz_i\bar{z}_i^{d-1}(1-\bar{z}_i^d)^{2c-1}\log(1-\bar{z}_i^d)}{x_i\left[1-(1-\bar{z}_i^d)^c\right]^2}\\&\quad+(\lambda-1)\sum_{i=1}^{n}\frac{cdz_i\bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}\log(1-\bar{z}_i^d)}{x_i\left[1-(1-\bar{z}_i^d)^c\right]}-\sum_{i=1}^{n}\frac{dz_i\bar{z}_i^{d-1}}{x_i(1-\bar{z}_i^d)},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial c\partial\theta}&=(\lambda-1)\sum_{i=1}^{n}\frac{dz_i\bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]}+(\lambda-1)\sum_{i=1}^{n}\frac{cdz_i\bar{z}_i^{d-1}(1-\bar{z}_i^d)^{2c-1}\log(1-\bar{z}_i^d)}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]^2}+\\&\quad(\lambda-1)\sum_{i=1}^{n}\frac{cdz_i\bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}\log(1-\bar{z}_i^d)}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]}-\sum_{i=1}^{n}\frac{dz_i\bar{z}_i^{d-1}}{x_i^2(1-\bar{z}_i^d)},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial d^2}&=-\frac{n}{d^2}-(c-1)\sum_{i=1}^{n}\frac{\bar{z}_i^{2d}\log(\bar{z}_i)^2}{(1-\bar{z}_i^d)^2}-(c-1)\sum_{i=1}^{n}\frac{\bar{z}_i^d\log(\bar{z}_i)^2}{1-\bar{z}_i^d}\\&\quad-(\lambda-1)\sum_{i=1}^{n}\frac{c^2\bar{z}_i^{2d}(1-\bar{z}_i^d)^{2(c-1)}\log(\bar{z}_i)^2}{\left[1-(1-\bar{z}_i^d)^c\right]^2}-(\lambda-1)\sum_{i=1}^{n}\frac{c(c-1)\bar{z}_i^{2d}(1-\bar{z}_i^d)^{c-2}\log(\bar{z}_i)^2}{1-(1-\bar{z}_i^d)^c}\\&\quad+(\lambda-1)\sum_{i=1}^{n}\frac{c\bar{z}_i^d(1-\bar{z}_i^d)^{c-1}\log(\bar{z}_i)^2}{1-(1-\bar{z}_i^d)^c}\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 \ell}{\partial d \partial \alpha} &= \sum_{i=1}^{n} \frac{z_i}{x_i \bar{z}_i} - (c-1)\sum_{i=1}^{n} \frac{z_i \bar{z}_i^{d-1}}{x_i(1-\bar{z}_i^d)} \\
&- (c-1)\sum_{i=1}^{n} \frac{d z_i \bar{z}_i^{2d-1}\log(\bar{z}_i)}{x_i(1-\bar{z}_i^d)^2} - (c-1)\sum_{i=1}^{n} \frac{d z_i \bar{z}_i^{d-1}\log(\bar{z}_i)}{x_i(1-\bar{z}_i^d)} \\
&+ (\lambda-1)\sum_{i=1}^{n} \frac{c z_i \bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}}{x_i\left[1-(1-\bar{z}_i^d)^c\right]^2} - (\lambda-1)\sum_{i=1}^{n} \frac{c^2 d z_i \bar{z}_i^{2d-1}(1-\bar{z}_i^d)^{2(c-1)}\log(\bar{z}_i)}{x_i\left[1-(1-\bar{z}_i^d)^c\right]^2} \\
&- (\lambda-1)\sum_{i=1}^{n} \frac{c(c-1) d z_i \bar{z}_i^{2d-1}(1-\bar{z}_i^d)^{c-2}\log(\bar{z}_i)}{x_i\left[1-(1-\bar{z}_i^d)^c\right]} \\
&+ (\lambda-1)\sum_{i=1}^{n} \frac{c d z_i \bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}\log(\bar{z}_i)}{x_i\left[1-(1-\bar{z}_i^d)^c\right]},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \ell}{\partial d \partial \theta} &= \sum_{i=1}^{n} \frac{z_i}{x_i^2 \bar{z}_i} - (c-1)\sum_{i=1}^{n} \frac{z_i \bar{z}_i^{d-1}}{x_i^2(1-\bar{z}_i^d)} \\
&- (c-1)\sum_{i=1}^{n} \frac{d z_i \bar{z}_i^{2d-1}\log(\bar{z}_i)}{x_i^2(1-\bar{z}_i^d)^2} - (c-1)\sum_{i=1}^{n} \frac{d z_i \bar{z}_i^{d-1}\log(\bar{z}_i)}{x_i^2(1-\bar{z}_i^d)} + \\
&(\lambda-1)\sum_{i=1}^{n} \frac{c z_i \bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]^2} - (\lambda-1)\sum_{i=1}^{n} \frac{c^2 d z_i \bar{z}_i^{2d-1}(1-\bar{z}_i^d)^{2(c-1)}\log(\bar{z}_i)}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]^2} - \\
&(\lambda-1)\sum_{i=1}^{n} \frac{c(c-1) d z_i \bar{z}_i^{2d-1}(1-\bar{z}_i^d)^{c-2}\log(\bar{z}_i)}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]} \\
&+ (\lambda-1)\sum_{i=1}^{n} \frac{c d z_i \bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}\log(\bar{z}_i)}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \ell}{\partial \alpha^2} &= -(d-1)\sum_{i=1}^{n} \frac{z_i^2}{x_i^2 \bar{z}_i^2} - (d-1)\sum_{i=1}^{n} \frac{z_i}{x_i^2 \bar{z}_i} - (c-1)\sum_{i=1}^{n} \frac{d^2 z_i^2 \bar{z}_i^{2(d-1)}}{x_i^2(1-\bar{z}_i^d)^2} \\
&- (c-1)\sum_{i=1}^{n} \frac{d(d-1) z_i^2 \bar{z}_i^{d-2}}{x_i^2(1-\bar{z}_i^d)} + (c-1)\sum_{i=1}^{n} \frac{d z_i \bar{z}_i^{d-1}}{x_i^2(1-\bar{z}_i^d)} \\
&- (\lambda-1)\sum_{i=1}^{n} \frac{c^2 d^2 z_i^2 \bar{z}_i^{2(d-1)}(1-\bar{z}_i^d)^{2(c-1)}}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]^2} \\
&- (\lambda-1)\sum_{i=1}^{n} \frac{c(c-1) d^2 z_i^2 \bar{z}_i^{2(d-1)}(1-\bar{z}_i^d)^{c-2}}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]} \\
&+ (\lambda-1)\sum_{i=1}^{n} \frac{c d(d-1) z_i^2 \bar{z}_i^{d-2}(1-\bar{z}_i^d)^{c-1}}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]} \\
&- (\lambda-1)\sum_{i=1}^{n} \frac{c d z_i \bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]} - \sum_{i=1}^{n} \frac{1}{x_i^4\left(\frac{\alpha}{x_i^2}+\frac{\theta}{x_i^3}\right)^2},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \ell}{\partial \alpha \partial \theta} &= -(d-1)\sum_{i=1}^{n} \frac{z_i^2}{x_i^3 \bar{z}_i^2} - (d-1)\sum_{i=1}^{n} \frac{z_i}{x_i^3 \bar{z}_i} \\
&- (c-1)\sum_{i=1}^{n} \frac{d^2 z_i^2 \bar{z}_i^{2(d-1)}}{x_i^3(1-\bar{z}_i^d)^2} - (c-1)\sum_{i=1}^{n} \frac{d(d-1) z_i^2 \bar{z}_i^{d-2}}{x_i^3(1-\bar{z}_i^d)} \\
&+ (c-1)\sum_{i=1}^{n} \frac{d z_i \bar{z}_i^{d-1}}{x_i^3(1-\bar{z}_i^d)} - (\lambda-1)\sum_{i=1}^{n} \frac{c^2 d^2 z_i^2 \bar{z}_i^{2(d-1)}(1-\bar{z}_i^d)^{2(c-1)}}{x_i^3\left[1-(1-\bar{z}_i^d)^c\right]^2} \\
&- (\lambda-1)\sum_{i=1}^{n} \frac{c(c-1) d^2 z_i^2 \bar{z}_i^{2(d-1)}(1-\bar{z}_i^d)^{c-2}}{x_i^3\left[1-(1-\bar{z}_i^d)^c\right]} \\
&+ (\lambda-1)\sum_{i=1}^{n} \frac{c d(d-1) z_i^2 \bar{z}_i^{d-2}(1-\bar{z}_i^d)^{c-1}}{x_i^3\left[1-(1-\bar{z}_i^d)^c\right]} \\
&- (\lambda-1)\sum_{i=1}^{n} \frac{c d z_i \bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}}{x_i^3\left[1-(1-\bar{z}_i^d)^c\right]} - \sum_{i=1}^{n} \frac{1}{x_i^5\left(\frac{\alpha}{x_i^2}+\frac{\theta}{x_i^3}\right)^2},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \ell}{\partial \theta^2} &= -(d-1)\sum_{i=1}^{n} \frac{z_i^2}{x_i^4 \bar{z}_i^2} - (d-1)\sum_{i=1}^{n} \frac{z_i}{x_i^4 \bar{z}_i} \\
&- (c-1)\sum_{i=1}^{n} \frac{d^2 z_i^2 \bar{z}_i^{2(d-1)}}{x_i^4(1-\bar{z}_i^d)^2} - (c-1)\sum_{i=1}^{n} \frac{d(d-1) z_i^2 \bar{z}_i^{d-2}}{x_i^4(1-\bar{z}_i^d)} \\
&+ (c-1)\sum_{i=1}^{n} \frac{d z_i \bar{z}_i^{d-1}}{x_i^4(1-\bar{z}_i^d)} - (\lambda-1)\sum_{i=1}^{n} \frac{c^2 d^2 z_i^2 \bar{z}_i^{2(d-1)}(1-\bar{z}_i^d)^{2(c-1)}}{x_i^4\left[1-(1-\bar{z}_i^d)^c\right]^2} \\
&- (\lambda-1)\sum_{i=1}^{n} \frac{c(c-1) d^2 z_i^2 \bar{z}_i^{2(d-1)}(1-\bar{z}_i^d)^{c-2}}{x_i^4\left[1-(1-\bar{z}_i^d)^c\right]} \\
&+ (\lambda-1)\sum_{i=1}^{n} \frac{c d(d-1) z_i^2 \bar{z}_i^{d-2}(1-\bar{z}_i^d)^{c-1}}{x_i^4\left[1-(1-\bar{z}_i^d)^c\right]} - (\lambda-1)\sum_{i=1}^{n} \frac{c d z_i \bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}}{x_i^4\left[1-(1-\bar{z}_i^d)^c\right]} \\
&- \sum_{i=1}^{n} \frac{1}{x_i^6\left(\frac{\alpha}{x_i^2}+\frac{\theta}{x_i^3}\right)^2}.
\end{aligned}
$$

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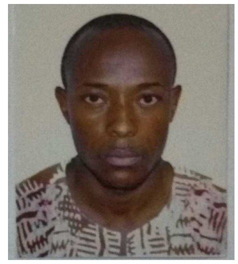

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