MACHAKOS UNIVERSITY

University Examinations 2021/2022 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

THIRD YEAR FIRST SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS)

SMA 304: NUMBER THEORY

DATE: 29/8/2022 **TIME: 8.30-10.30 AM**

INSTRUCTION:

Answer Question One and Any Other Two Questions

QUESTION ONE - (30 MARKS)

a) Using appropriate examples, distinguish between the meaning of the term *congruence modulo* m and the term *reduced residue modulo* m. (5 marks)

b) Let x, y and z be integers. Show that if $x \mid y$ and $y \mid z$ then $x \mid z$. (5 marks)

c) Let $a \in \mathbb{Z}$ and $b_1, b_2, \ldots, b_n$ be a sequence of integers such that $a \mid b_i$ for each $i = 1,2, \ldots, n$. Prove that

$$a \mid (b_1y_1 + b_2y_2 + \cdots + b_ny_n)$$

for any $y_1, y_2, \ldots, y_n \in \mathbb{Z}$. (5 marks)

d) Prove that if q is an odd integer then $q^2 = 8k + 1$ for some $k \in \mathbb{Z}$. (5 marks)

e) Express the following pairs of integers in the form $a = bq + r$ with $0 \leq r < b$

i. $a = -3561, b = 98,$ (5 marks)

ii. $a = 337, b = 751.$ (5 marks)

QUESTION TWO (20 MARKS)

a) Let $i \in \mathbb{Z}$. Prove that either $i^2 = 4j$ or $i^2 = 4j + 1$ for some integer j. (4 marks)

b) Explain the meaning of the *Euler* ϕ*-function* of a positive integer n hence or otherwise determine $\phi(15)$. (5 marks)

c) Using the Euclidean algorithm, express $\gcd(2327,819)$ in the form $d = 2327x + 819y$. (6 marks)

d) Suppose that $a, b, n \in \mathbb{Z}$ with $n > 0$. Given that if $a \equiv b(\text{mod } n)$, Prove that $a^m \equiv b^m(\text{mod } n)$ for any positive integer m. (5 marks)

QUESTION THREE (20 MARKS)

a) Let a, b, m be integers with $m > 0$ and suppose that $a \equiv b(mod \quad m)$. Prove that $ax \equiv bx(mod \quad mx)$ for any integer x. (4 marks)

b) Prove that if y is an integer then $3 \mid (y^3 - y)$. (5 marks)

c) Solve the linear congruence $133x - 6 \equiv 32(\text{mod } 209)$ (7 marks)

d) Construct at least two distinct complete residue systems modulo 8. For each of the given complete residue systems, determine their corresponding reduced residue systems modulo 8. (4 marks)

QUESTION FOUR (20 MARKS)

a) Suppose that a, b are non-zero integers such that $a = bq + r$ for some integers q, r. Prove that if $d = \gcd(b, r)$ then $d \mid a$. (7 marks)

b) Suppose that $a \in \mathbb{Z}$ is a positive integer greater than 1. Prove that the number $\frac{a^3+5a}{3}$ is an integer. (6 marks)

c) Suppose that a, b are non-zero integers. Prove that $\gcd(6a - 9b, 9a - 13b) \mid b$. Hence or otherwise show that $\gcd(6a - 9, 9a - 13) = 1$ for any integer a. (7 marks)

QUESTION FIVE (20 MARKS)

a) Distinguish between *linear* and non-linear Diophantine equations. (2 marks)

b) Let a, b, c, d, m be non-zero integers with $m > 0$. Suppose that $a \equiv b(\text{mod} m)$ and $c \equiv d(\text{mod} m)$. Prove that

   i. $a - cn \equiv b - dn(\text{mod} m)$ for any integer n, (4 marks)

   ii. $ac - z \equiv bd - z(\text{mod} m)$ for any integer z. (4 marks)

c) Solve the following linear Diophantine Equations (4 marks)

   i. $111x + 321y = 690$

   ii. $54x + 21y = 91$

d) Prove that if a is an integer neither divisible by 2 nor 3 then $a^2 \equiv 1(\text{mod } 24)$. (6 marks)