MACHAKOS UNIVERSITY

University Examinations 2021/2022 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR FIRST SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)

BACHELOR OF SCIENCE (STATISTICS AND PROGRAMMING)

BACHELOR OF SCIENCE (MATHEMATICS)

SMA 402: FIELD THEORY

DATE: 29/8/2022 **TIME: 2.00-4.00 PM**

INSTRUCTION:

Answer Question One and Any Other Two

QUESTION ONE - (30 MARKS)

a) Let F, K be fields. Explain the meaning of the following terms:

i. a *field extension* F/K, (3 marks)

ii. an *algebraic element* $\alpha \in F$ over K, (4 marks)

b) Using relevant examples distinguish between a field of positive characteristic p and a field of characteristic 0. (5 marks)

c) Consider the polynomials $f(x) = 2x^5 + 3x^4 + 2$ and $g(x) = 2x^9 + x^4$ in $\mathbb{Z}_4[x]$.

i. Compute the sum $f(x) + g(x)$. (3 marks)

ii. Determine the degree of the product $f(x)g(x)$. (4 marks)

d) Describe the splitting field of the polynomial $f(x) = x^4 + 1$ over $\mathbb{Q}$. (6 marks)

e) Prove that the number $\sqrt[2]{3} + 1$ is algebraic over $\mathbb{Q}$. (5 marks)

QUESTION TWO (20 MARKS)

a) Suppose that F/K is a field extension and $\alpha \in F$ is algebraic over K. Let $f \in K[x]$ be a non-constant polynomial of the least degree such that $f(\alpha) = 0$. Prove that

i. $f(x)$ is irreducible over K. (5 marks)

ii. if $p(x) \in K[x]$ is a polynomial such that $p(\alpha) = 0$ then $f(x)|p(x)$. (5 marks)

b) Let F be any field. Prove that for any $n > 0$ then $g(x) = x - 1$ is a divisor of $f(x) = x^n - 1$ in the polynomial ring $F[x]$. (5 marks)

c) Let $\alpha = \sqrt{2} + i \in \mathbb{C}$. Construct the minimal polynomial of α over $\mathbb{Q}$ hence or otherwise determine a basis for the field extension $\mathbb{Q}(\alpha)/\mathbb{Q}$. (5 marks)

QUESTION THREE (20 MARKS)

a) Distinguish between an *algebraically closed field* K and an *algebraic closure* $\overline{K}$ of a field K. (2 marks)

b) Determine the root of $4x + 1$ in $\mathbb{Z}_5[x]$. (2 marks)

c) Let $K \subset L \subset F$ be fields. Show that if $\alpha \in F$ is algebraic over L and L/K is a finite extension then α is algebraic over K. (6 marks)

d) Construct the minimal polynomial and the basis of the simple algebraic extension $\mathbb{Q}(\sqrt[3]{2} + \sqrt{2})$ over $\mathbb{Q}$. (10 marks)

QUESTION FOUR (20 MARKS)

a) Let $f(x) \in K[x]$ be a non-constant polynomial. Explain the meaning of the *splitting field* of $f(x)$ over K. (4 marks)

b) Let F be any field. Prove that any polynomial $f(x) \in F[x]$ of degree 2 or 3 is irreducible if and only if $f(x)$ has no roots in F. (4 marks)

c) Describe the field extension $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$ where $p \in \mathbb{Q}$ is square free. (6 marks)

d) Suppose that E/F is an extension satisfying $[E:F] = p$ where p is a prime number. Prove that E/F contains no proper intermediate field. (6 marks)

QUESTION FIVE (20 MARKS)

a) Let F/K be a field extension. Show that $[F:K] = 1$ if and only if $F = K$. (5 marks)

b) Suppose that $K_1, K_2, \ldots, K_n$ and are subfields of a field K. (5 marks)

   i. Prove that $K_1 \cap K_2 \cap \ldots \cap K_n$ is a subfield of K.

   ii. Describe the meaning of a *prime* subfield of a field K.

c) Suppose that F/L and L/K are field extensions such that $[F:L] = n$ and $[L:K] = m$. Prove that F/K is an extension such that $[F:K] = nm$. (10 marks)