MACHAKOS UNIVERSITY

University Examinations 2021/2022 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR FIRST SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS)

SMA 404: COMPLEX ANALYSIS II

DATE: 30/8/2022 **TIME: 2.00-4.00 PM**

INSTRUCTION:

Answer Question One and Any Other Two Questions

QUESTION ONE (30 MARKS)

a) State without proof the *Cauchy's integral theorem*. (2 marks)

b) Determine the poles of the function $f(z) = \frac{z+2i}{z^2(z^2+4)}$ stating clearly the order of each pole.

(4 marks)

c) Use the Cauchy's residue theorem to evaluate the following integral

$$\oint_C \frac{2z-i}{z^2(z-2)}dz$$

where C is the circle $|z| = 3$. (5 marks)

d) Consider the transformation defined by $w = z^2 + 1$. Compute:

i. the coefficient of magnification at $z_0 = 1 + 3i$, (4 marks)

ii. the angle of rotation at $z_0 = 1 + 3i$, (4 marks)

iii. the critical points of the transformation. (2 marks)

e) Consider the function $u(x, y) = y^3 + ax^2y$

i. Determine the constant a such that $u(x, y)$ is a harmonic function. (3 marks)

ii. Use the *harmonic conjugate theorem* to determine the harmonic conjugate of the $u(x, y)$. (6 marks)

QUESTION TWO (20 MARKS)

a) Using examples, distinguish between a *Conformal transformation*, and an *Isogonal transformation*. (4 marks)

b) Determine whether the function $f(z) = (z + i)^2$ satisfy the Schwarz reflection principle. (6 marks)

c) Compute the residues of the function $f(z) = \frac{1}{z^2(z+1)}$ at all of its poles. (4 marks)

d) Compute the Cauchy's principal value of the following indefinite integral

$$\int_{\infty}^{\infty} \frac{x^2}{(x^2 + 4)^2} dx.$$

(6 marks)

QUESTION THREE (20 MARKS)

a) Explain the meaning of the term *analytic continuation*. (2 marks)

b) Compute the Taylor Series for the function

$$f(z) = \cos(z)$$

about the point $a = \frac{\pi}{4}$. (4 marks)

c) Evaluate the integral $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$. (4 marks)

d) Prove that the series $g(z) = \frac{1}{1+i} \sum_{n=0}^{\infty} \left(\frac{z+i}{1+i}\right)^n$ is an analytic continuation of the function $f(z) = \left(\frac{1}{1-z}\right) = \sum_{n=0}^{\infty} z^n$ and sketch the region where the two functions are analytic. (7 marks)

e) State without proof the Schwarz reflection principle. (3 marks)

QUESTION FOUR (20 MARKS)

a) Explain the meaning of the following terms:

i. *harmonic function* $h(x, y)$, (2 marks)

ii. *harmonic conjugate* of a function $h(x, y)$. (2 marks)

b) Evaluate the contour integral $\oint_C \frac{(z-1)}{z(z+1)(4z-15)} dz$ where C is given to be the circle $|z| = 2$. (5 marks)

c) Determine the Laurent series for the function

$$f(z) = \frac{1}{1+z}$$

which are valid where $1 < |z| < \infty$. (5 marks)

d) Prove that for closed polygons, the sum of the exponents

$$\frac{\alpha_1}{\pi} - 1, \frac{\alpha_2}{\pi} - 1, \cdots \frac{\alpha_n}{\pi} - 1$$

in the Schwarz-Christoffel transformation is -2. (6 marks)

QUESTION FIVE (20 MARKS)

a) Let $z_0 = r_0 e^{i\theta_0}$ be a fixed point on $\mathbb{C}$ and $n > 0$ be an integer. Consider the transformation $w = z^n$.

i. Show that the angle of rotation at z_0 is $(n-1)\theta_0$. (2 marks)

ii. Compute the coefficient of magnification at z_0. (2 marks)

b) Consider the transformation given by $w = z^2$.

i. Determine the image of the triangular region on the z-plane bounded by the lines $x = 0, y = 0$ and $x + y = 1$. (5 marks)

ii. Verify conformality at the point $z = 1$ by considering the intersection of the lines $y = 0$ and $x + y = 1$. (4 marks)

c) State without proof the *Schwarz-Christoffel transformation* theorem. (3 marks)

d) If $w = \infty$ is mapped onto a vertex z_n on the z-plane, prove that the term containing $(w - u_n)$ in the Schwarz-Christoffel transformation can be replaced by 1. (4 marks)