MACHAKOS UNIVERSITY

University Examinations 2021/2022 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR FIRST SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)

BACHELOR OF SCIENCE (STATISTICS AND PROGRAMMING)

BACHELOR OF SCIENCE (MATHEMATICS)

SMA 437: BIOMATHEMATICS

DATE: 22/8/2022 **TIME: 8.30-10.30 AM**

INSTRUCTION:

Answer Question One and Any Other Two Questions

QUESTION ONE (30MARKS) COMPULSORY

a) A livestock farmer in lower Eastern Kenya has a ranch that can support a flock of 500 goats, which he sells to slaughter houses in the locality. The population of the goats grows at a rate of 30% per year.

i. What is the maximum number of goats that can be sold out each year sustainably? (4 marks)

ii. In order to maintain the herd size at 250 goats, how many goats should be sold each year? What should be the minimum size of the initial flock for this to happen? (4 marks)

b) The non-dimensional predator-prey model is given $\begin{matrix} \dot{u} = u(1-u) \\ \dot{v} = \alpha v(u-1) \end{matrix}$ where α is a positive constant. Given that the phase trajectories for this model are $\alpha u + v - \ln(u^{\alpha} v) = C$ where C is a constant. Sketch the phase trajectories on the uv phase plane and briefly explain the dynamics (6 marks)

c) Given the following form of the SIR model;

$$\dot{S} = -\frac{\beta SI}{N}, \ \ \beta > 0,$$
$$\dot{I} = \frac{\beta SI}{N} - \sigma I, \sigma > 0,$$
$$\dot{R} = \sigma I,$$

show that $S - \frac{N}{R_0}\ln S + I = C$ where $N = S + I + R$ is the total population, $R_0 = \frac{\beta}{\sigma}$ is the reproduction number and C is a constant of integration. (6 marks)

d) The spread of common cold/flu in a given confined population can be modeled using the SIR model:

$$\dot{S} = -\beta SI, \ \ \beta > 0, S(0) > 0$$
$$\dot{I} = \beta SI - \sigma I, \sigma > 0, I(0) > 0$$
$$\dot{R} = \sigma I, \ R(0) = 0$$

i. Sketch the flow diagram used to deduce the above model equations. (4 marks)

ii. State any three assumptions used to develop the above SIR model that describes the spread of the common cold/flu. (3 marks)

iii. Realistically, the population size changes and common cold is known to recur i.e. a recovered individual can get the disease once again. Improve the given model equations by accounting for this realistic phenomenon. (3 marks)

QUESTION TWO (20 MARKS)

a) In a certain study, the spread of malaria in the human (h) and mosquito (v) population is modeled as given below;

$$\dot{S}_h = \mu_h - \frac{bm\beta_h S_h I_v}{N_h} - \mu_h S_h\,,$$
$$\dot{I}_h = \frac{bm\beta_h S_h I_v}{N_h} + \frac{\sigma bm\beta_h (N_h - S_h - I_h) I_v}{N_h} - \alpha_1 I_h\,,$$
$$\dot{N}_h = \mu_h - \mu_h N_h - \delta_h I_h\,,$$
$$\dot{I}_v = \frac{b\beta_v (1 - I_v) I_h}{N_h} - \mu_v I_v$$

Where all constants are positive. Given that the endemic equilibrium is $E^* = (S_h^*, I_h^*, N_h^*, I_v^*)$, determine the expressions of S_h^*, N_h^*, I_v^* in terms of I_h^*. (10 marks)

b) The Lokta-Volterra model for two-species ecosystem is given by: $\begin{aligned}\dot{x} &= ax - cxy \\ \dot{y} &= -by + dxy\end{aligned}$ where a, b, c, d are positive constants.

i. Determine the equilibrium points of the model (4 marks)

ii. Determine the phase paths of this system of equations. (6 marks)

QUESTION THREE (20 MARKS)

a) In a given bacterial population of size $N(t)$, the nutrient-depletion can be modeled as $\frac{dN}{dt} = k(C_0 - \alpha N)N$ where α represents the units of nutrient consumed to produce one unit of population increment. C_0 is the initial amount of nutrient available for the population.

i. Show that the model equation can be written in the form $\frac{dN}{\left(1-\frac{N}{B}\right)N} = rdt$ where $r = kC_0$ and $B = \frac{C_0}{\alpha}$ (3 marks)

ii. Solve the equation in i) above and show that for $t \to \infty$ the population approaches B (7 marks)

b) Given the dynamical system:
$$\begin{aligned}\dot{x} &= y(1-x^2)\\ \dot{y} &= -x(1-y^2)\end{aligned}$$

i. Determine the equilibrium points (5 marks)

ii. Show that the phase paths are given by the equation $(1-x^2)(1-y^2) = C$, where C is a constant. (5 marks)

QUESTION FOUR (20 MARKS)

a) Given the predator, $P(t)$-prey, $N(t)$ model;

$$\dot{N} = N\left[r\left(1-\frac{N}{K}\right) - \frac{kP}{N+D}\right]$$
$$\dot{P} = P\left[s\left(1-\frac{hP}{N}\right)\right]$$

where r, K, k, D, s, h are positive constants.

i. Determine the corresponding non-dimensional model equations which have only the 3 parameters $a = \frac{k}{hr}, b = \frac{s}{r}, d = \frac{D}{K}$. Take $P(t) = \frac{Kv(\tau)}{h}, \tau = rt$ (7 marks)

ii. For $a = 4, d = -2$ find the equilibrium points of the non-dimensional model. (6 marks)

b) Consider the predator, $P(t)$-prey, $N(t)$ model $\begin{array}{l}\dot{N} = N[a - bP] \\ \dot{P} = P[cN - d]\end{array}$ where a, b, c, d are positive constants. Improve this model such that each of the species $P(t)$ and $N(t)$ have logistic growth in the absence of the other and state the meaning of each constant in the improved model. (7 marks)

QUESTION FIVE (20 MARKS)

The SEIR model consists of the following equations;

$$
\begin{aligned}
\dot{S} &= -\beta SI + \lambda - \mu S \\
\dot{E} &= \beta SI - (\mu + k)E \\
\dot{I} &= kE - (\gamma + \mu)I \\
\dot{R} &= \gamma I - \mu R
\end{aligned}
$$

a) Sketch the flow diagram of the above model equations. (5 marks)

b) State what each of the following parameters β, λ, μ, k, γ represents. (5 marks)

c) Obtain the matrices F and V^{-1} for this model, hence show that the reproduction number is given by $R_0 = \dfrac{k\beta\lambda}{\mu(k + \mu)(\gamma + \mu)}$ (10 marks)