MACHAKOS UNIVERSITY

University Examinations 2021/2022 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

THIRD YEAR FIRST SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)

BACHELOR OF SCIENCE (MATHEMATICS)

BACHELOR OF ECONOMICS

BACHELOR OF EDUCATION

SMA 300: REAL ANALYSIS

DATE: 1/9/2022 **TIME: 2.00-4.00 PM**

INSTRUCTION:

Answer Question One and Any Other Two Questions

QUESTION ONE (30 MARKS)

a) Describe what is meant by the statement "The set $\mathbb{R}$ of real numbers is complete". Exhibit an example to illustrate why the set $\mathbb{Q}$ of rational numbers is not complete. (3 marks)

b) Explain the following terms

i. An upper bound,

ii. Infimum of a set

iii. Countable set. (3 marks)

c) Determine whether the following sequence is monotonic and/or bounded. If bounded determine its supremum and infimum.

$\left\{\frac{2}{n^2}\right\}_{n=5}^{\infty}$ (5 marks)

d) Show that if $p \in A$ and $p \notin A'$, then p is an isolated point (4 marks)

e) Let A and B be neighborhoods of a point x. Show that the intersection $A \cap B$ is a neighborhood of x (3 marks)

f) Prove that if a sequence converges then its limit is unique. (4 marks)

g) Prove that a set $X$is closed iff its complement is open (5 marks)

h) Determine and prove whether the set of all integers is countable. (3 marks)

QUESTION TWO (20 MARKS)

a) Prove that every convergence sequence is a Cauchy sequence. (4 marks)

b) Prove that the set of rational numbers $\mathbb{Q}$ has no interior point (4 marks)

c) Consider the set $A = (1,10) \cup \{11\}$ .Show that 11 is the isolated point of A and both 1 and 10are the limit points of A (5 marks)

d) Let $X = \mathbb{R}$. Describe what is meant by

i. Neighborhood of a point $x \in X$. (2 marks)

ii. An open set $O \in X$. (2 marks)

e) Given that $A = (3,7)$ find the exterior and the boundary of A (3 marks)

QUESTION THREE (20 MARKS)

a) Define what is meant by a convergent sequence hence show from first principles that the sequence: $y_n = 3 + \frac{(-1)^n}{n^2}$ converges to 3 in $\mathbb{R}$. (4 marks)

b) Determine the n^th^ partial sum of the series below. Hence test for its convergence.

$\sum_{n=1}^{\infty} \frac{3}{n^2+3n+2}$ (5 marks)

c) Show that $f(x) = 2x + 1$ is Riemann integrable on $[1,2]$ and hence find $\int_1^2 (2x + 1)dx$, (8 marks)

d) Show that a finite intersection of open sets is open. (3 marks)

QUESTION FOUR (20 MARKS)

a) Explain the following terms

i. Monotonic increasing sequence. (1 mark)

ii. Cauchy sequence. (2 marks)

b) Show that the sequence $\left(\frac{1}{2^n}\right)$ in $\mathbb{R}$ is a Cauchy sequence (3 marks)

c) Use comparison test determine the convergence of

$\sum_{n=2}^{\infty} \frac{n^2+2}{n^4+5}$ (3 marks)

d) Use ratio test determine the convergence of

$\sum_{n=2}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$ (5 marks)

e) For each positive integer k, let $\mathbb{R}^k$ be the set of all k-tuples where $X = \{x_1, \dots x_k\}$. Define a function d by $d(x,y) = \sum_{i=1}^{k} |x_i - y_i| \; \forall x, y \in \mathbb{R}$. Show that d is a metric space.

(6 marks)

QUESTION FIVE (20 MARKS)

a) Prove that convergence of a sequence implies boundedness. Use an example to show that the converse need not be true. (5 marks)

b) Determine the convergence or divergence of

$\sum_{n=1}^{\infty} \frac{4\sqrt{n}-1}{n^2+2\sqrt{n}}$ (5 marks)

c) Prove that a finite intersection of open sets is open (5 marks)

d) Prove that if $A, B \subset \mathbb{R}$, then $(A \cup B)' = A' \cup B'$ (5 marks)