MACHAKOS UNIVERSITY

University Examinations 2021/2022 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR FIRST SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)

BACHELOR OF SCIENCE (MATHEMATICS)

BACHELOR OF EDUCATION

SMA 403: GALOIS THEORY

DATE: 30/8/2022 **TIME: 8.30-10.30 AM**

INSTRUCTION:

Answer Question One and Any Other Two Questions

QUESTION ONE (30 MARKS) (COMPULSORY)

a) Let $f(x) \in \mathbb{Q}[x]$. A primitive root of $f(x) = 0$ is $x = \sqrt{7} + \sqrt{11}$.

i. Identify the splitting field of $f(x)$. (2 marks)

ii. Determine $f(x)$ as a minimal polynomial of this root. (3 marks)

iii. Identify the Galois group of $f(x)$. (3 marks)

iv. Let K be a field extension of afield H. Define the Galois group $G(K/H)$. (3 marks)

b) Define what is meant by a solvable group G, hence show that the Symmetric group S_4 is solvable. (4 marks)

c) Let Ebe a field of characteristic different from 2.

i. Show that if $f(x) = Ax^4 + Bx^2 + C$ where $A \neq 0$, then $f(x)$ is solvable by radicals over E . (4 marks)

ii. If $A = 1$, $b = -16$, $C = 4$ determine the Galois group of $f(x)$ over E in (i). (4 marks)

d) "There is no general formula for solving a fifth-degree polynomial equation''. Discuss. (7 marks)

QUESTION TWO (20 MARKS)

a) Briefly state the fundamental theorem of Galois theory. (3 marks)

b) Let $f(x) = x^4 - 4x^2 - 5$ over $\mathbb{Q}$.

i. Find the zeros of the polynomial $f(x)$. (4 marks)

ii. Find the splitting field K of the polynomial $f(x)$ and state $[K:\mathbb{Q}]$. (3 marks)

iii. Construct the Galois group $G(K/\mathbb{Q})$ (4 marks)

c) By using (b) above construct lattice diagrams of the subgroups of Galois group and subfields of the splitting field. (4 marks)

d) Identify with reasons a subgroup of the symmetric group S_4 isomorphic to $G(K/\mathbb{Q})$. (2 marks)

QUESTION THREE (20 MARKS)

a) Let$(\alpha_i, i \in \mathbb{N})$be a collection of automorphism of a field K, show $K\{\alpha_i\} = \{a \in K: \alpha_i(a) = a \, \forall \, \alpha_i\}$ is a subfield of K. (4 marks)

b) Let $H = \mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{7})$. Consider the Galois group $G(H/\mathbb{Q})$. The field H may be considered as a vector space. Identify the elements of the basis $\mathscr{B}$ of this vector space and state its dimension. (5 marks)

c) Let $\tau_3, \tau_5, \tau_7 \in G(H/\mathbb{Q})$ such that $\tau_3: \sqrt{3} \mapsto -\sqrt{3}, \tau_5: \sqrt{5} \mapsto -\sqrt{5}, \tau_7: \sqrt{7} \mapsto -\sqrt{7}$. Compute the following indicated elements of H.

i. $\tau_3\tau_5^2\tau_7\left(\sqrt{21} + \sqrt{15}\right)$ (3 marks)

ii. $\tau_7\tau_5\tau_3\left(\frac{\sqrt{3}-\sqrt{5}}{\sqrt{7}-\sqrt{21}}\right)$ (3 marks)

d) Apart from τ_3, τ_5, τ_7 identify and describe the remaining elements of $G(H/\mathbb{Q})$. (5 marks)

QUESTION FOUR (20 MARKS)

a) Define what is meant by a primitive element of a Field extension E of F. Identify the primitive element of the extension $F = \mathbb{Q}(\sqrt{2}, i)$ of $\mathbb{Q}$. Determine the minimal monic polynomial associated with the primitive element you have identified. (8 marks)

b) Give definitions of the following terms:

i. Separable extension of a field F (2 marks)

ii. Normal extension of a field F. (2 marks)

c) State the symmetric function theorem. (2 marks)

d) Express the following symmetric functions using elementary symmetric functions in x_1, x_2, x_3:

i. $x_1^2 + x_2^2 + x_3^2$ (3 marks)

ii. $x_1^3 + x_2^3 + x_3^3$ (3 marks)

QUESTION FIVE (20 MARKS)

a) Prove that the set of all automorphism of a field H is a group under composition of functions. (4 marks)

b) Let K be a field extension of H and $h(x)$ be a polynomial in $H[x]$. Show that any automorphism in $G(K/H)$ defines a permutation of the roots of $h(x)$ that lie in K. (4 marks)

c) Define what is meant by a cyclotomic extension for any field K. Hence or otherwise write down the splitting field of the polynomial $x^5 - 1 \in \mathbb{Q}[x]$ and its corresponding Galois Group.

(6 marks)

d) Use Cardan's Formula to solve the polynomial equation $x^3 + x^2 - 3x + 2 = 0$. (6 marks)