MACHAKOS UNIVERSITY

University Examinations 2021/2022 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR FIRST SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)

BACHELOR OF SCIENCE (MATHEMATICS)

SMA 436: METHODS OF APPLIED MATHEMATICS II

DATE: 2/9/2022 **TIME: 2.00-4.00 PM**

INSTRUCTION:

Answer Question One and Any Other Two Questions

QUESTION ONE (30 MARKS)

a) Given than the $\delta_j^i = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$ Expand $\delta_j^i x_i x_j$. (3 marks)

b) If a_{ij} are constants and $\delta_{pq} = \frac{\partial x_p}{\partial x_q}$, calculate the partial derivative $\frac{\partial}{\partial x_k}\left(a_{ij}x_i x_j\right)$ when the constants $a_{ij} = a_{ji}$. (3 marks)

c) Prove by tensor notation and the product rule for matrices that $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$ for any two conformal matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. (4 marks)

d) Define an Affine tensor and hence a Cartesian tensor. (2 marks)

e) Find the Euclidean metric tensor in matrix form for spherical coordinates given that spherical coordinates x^i are connected to rectangular coordinates via
$\bar{x}^1 = x^1 \sin x^2 \cos x^3$, $\bar{x}^2 = x^1 \sin x^2 \sin x^3$ and $\bar{x}^3 = x^1 \cos x^2$ (6 marks)

f) Define the Christoffel symbols of first and second kind. (2 marks)

g) Consider the initial value problem;

$$\frac{d^2y}{dx^2} + xy = 1 \,,$$
$$y(0) = 0\,, y'(0) = 0$$

Transform this initial value problem to a Volterra integral equation. (7 marks)

h) Define a singular integral equation. (3 marks)

QUESTION TWO (20 MARKS)

a) State the Quotient law. (2 marks)

b) A quantity $A(p,q,r)$ is such that in the coordinate system x^i, $\mathrm{A}(p,q,r)B_r^{qs} = C_p^s$, where B_r^{qs} is an arbitrary tensor and C_p^s is a known tensor. Prove that $A(p,q,r)$ is a tensor. (10 marks)

c) Calculate the Christoffel symbols of the second kind for the Euclidean metric in polar given by;

$$G = \begin{bmatrix} 1 & 0 \\ 0 & (x^1)^2 \end{bmatrix}$$ (8 marks)

QUESTION THREE (20 MARKS)

a) Suppose that in $\mathbb{R}^3$ a metric field is given in x^i by

$$g_{ij} = \begin{bmatrix} (x^1)^2 - 1 & 1 & 0 \\ 1 & (x^2)^2 & 0 \\ 0 & 0 & {}^{64}/_{9} \end{bmatrix} \text{Where } [(x^1)^2 - 1](x^2)^2 \neq 1$$

i. Show that if extended to all admissible coordinate systems according to the transformation law of covariant tensors, this matrix field is metric. (4 marks)

ii. For the metric compute the arc-length parameter and hence determine the length of the curve; $C: \begin{cases} x^1 = 2t - 1 \\ x^2 = 2t^2 \\ x^3 = t^3 \end{cases}$

For $0 \leq t \leq 1$ (8 marks)

b) Determine the resolvent kernel $\Gamma(x,\xi;\lambda)$ associated with $K(x,\xi) = 1 - 3x\xi$ in the interval $(0,1)$ in the form of the power series in $\boldsymbol{\lambda}$, obtaining the first three terms. (8 marks)

QUESTION FOUR (20 MARKS)

a) Given the special coordinate system x^i defined from rectangular coordinates $\bar{x}^i$ by

$x^1 = \bar{x}^1$ and $x^2 = e^{\bar{x}^2 - \bar{x}^1}$;

i. Compute the Euclidean metric tensor. (4 marks)

ii. Given that $C: x^1 = 3t, x^2 = e^t, 0 \le t \le 2$, calculate the length of the curve. (5 marks)

iii. Interpret a(i) above geometrically. (3 marks)

b) Consider the boundary value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0 \,,$$

$$y(0) = 0 \,, y(l) = 0$$

Transform this boundary value problem to a Fredholm equation of the second kind. (8 marks)

QUESTION FIVE (20 MARKS)

a) Using Green's function transform the following problem into Fredholm integral equations.

$$y'' + xy = 1$$

$$y(0) = 0, y(l) = 1$$

(8 marks)

b) Show that the Bessel equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda x^2 - 1)y = 0 \,,$$

$$y(0) = 0 \,, y(1) = 0$$

transforms to the integral equations

$$y(x) = \lambda \int_0^1 G(x, \xi)\xi y(\xi) d\xi$$

Where

$$G(x, \xi) = \begin{cases} \frac{x}{2\xi}(1 - \xi^2), x < \xi \\ \frac{\xi}{2x}(1 - x^2), x > \xi \end{cases}$$

(12 marks)