MACHAKOS UNIVERSITY

University Examinations for 2022/2023 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR SECOND SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (COMPUTER SCIENCE)

SMA 407: MEASURE AND INTEGRATION

DATE: **TIME:**

INSTRUCTION:

1. Answer **question one** and **any other two** questions
2. Write the name of the unit and registration number on each page of your answer sheet.

QUESTION ONE (COMPULSORY) (30 MARKS)

a) Give definitions of each of the following

i. Sequential cover (3 marks)

ii. Lebesgue outer measure (3 marks)

iii. Lebesgue measurable subset E of $\Re$ (2 marks)

b) Show that $\mu^*(\{x\}) = 0$ for each $x \in \Re$ (4 marks)

c) Show that for any Interval $\mu^*(I) = \lambda(I)$, where $\lambda(I)$ is the usual length of an interval in $\Re$

(4 marks)

d) Show that if E is a countable subset of $\Re$ then $\mu^*(E) = 0$ (4 marks)

e) Using (d) above show that the empty set $\varnothing$ is Lebesgue measurable. (4 marks)

f) Give the definition of a measure space. Let $L(X, \mathfrak{x}\, V)$ be a measure space. $G, H \in \mathfrak{x}$ with $G \subseteq H$. Show that if $V(G) < \infty$ then $V(G) \leq V(H)$. (6 marks)

QUESTION 2 (20 MARKS)

a) Let $A \in \sigma m$ with $\mu(A) < \infty$ Given that B is any subset of $\Re$ such that $B \supset A$ show that

$\mu^*(B - A) = \mu^*(B) - \mu(A)$ (5 marks)

b) Let $A \in \sigma m$ and $B \subset \Re$ show that $\mu^*(A \cup B) + \mu^*(A \cap B) = \mu(A) + \mu^*(B)$ (7 marks)

c) Show that if $A, B \in \sigma m$ then $A \cup B \in \sigma m$. (5 marks)

d) Deduce from (c) above that $A \cap B \in \sigma m$. (3 marks)

QUESTION 3 (20 MARKS)

a) Let μ^* be the Lebesgue outer measure on $\Re$

Prove the following:

i. $\mu^*(\emptyset) = 0$ (4 marks)

ii. For any $A \in \mathfrak{X}$ the set function $\mu_o^*: \mathfrak{X} \to \Re$ defined by

$$\mu_0^*(A) = \mu^*(A \cap C)$$

where C is a fixed set in $\mathfrak{X}$, is also an outer measure in $\Re$. (4 marks)

b) i) Let $E \subseteq \Re$ with $\mu^*(E) < \infty$. If there is a subset A of E such that A is Lebesgue measurable and $\mu(A) = \mu^*(E)$ show that E is also Lebesgue measurable. (3 marks)

ii) Let E be Lebesgue measurable with $\mu(E) = 0$

If $A \subseteq E$ show that A is Lebesgue measurable and $\mu(A) = 0$ (3 marks)

c) Give the definition of a Lebesgue non-measurable subset of $\Re$.

Show that if A is a Lebesgue non-measurable subset of $\Re$ then there is a subset E of A with $0 < \mu^*(E) < \infty$. (6 marks)

QUESTION FOUR (20 MARKS)

a) Let f be a function on $\Re$ defined by

$$f(x) = \begin{cases} \frac{1}{2}, & x \le 1 \\ |x|, & -1 < x < 1 \\ -1, & 1 \le x \end{cases}$$

Show that f is measurable. (6 marks)

b) Let $(X, \mathfrak{X})$ be a measurable space, $f: X \to \Re$ be $\mathfrak{X}-$ measurable and $\theta: \Re \to \Re$ be continuous function.

Show that the composite function $(\theta \circ f)$ defined by $(\theta \circ f)(x) = \theta(f(x)), \ \forall\, x \in X$ is $\mathfrak{X}-$ measurable. (8 marks)

c) Let χ_E be the characteristic function on the set E. Show that χ_E is measurable if and only if E is Lebesgue measurable. (6 marks)

QUESTION FIVE (20 MARKS)

a) Give the definition of an abstract measure space. Identify an example (4 marks)

b) Explain and give an example on what we mean by some property holding $\mu-$ almost everywhere ($\mu.a.e$) on any abstract measure space. (5 marks)

c) i) Give the definition of an abstract Integral. (3 marks)

ii) Define $f:[0,7] \to \Re$ by

$$f(x) = \begin{cases} 1\ ,\ 0 \le x < 2 \\ 3\ ,\ 2 \le x < 3 \\ 5\ ,\ 3 \le x < 5 \\ 9\ ,\ 5 \le x < 7 \end{cases}$$

Use the definition in c(i) above to show how you would construct the abstract integral hence or otherwise evaluate it. (8 marks)