MACHAKOS UNIVERSITY

University Examinations 2017/2018

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR SECOND SEMESTER EXAMINATION FOR BACHELOR OF EDUCATION (SCIENCE)

SMA 409: ALGEBRAIC GEOMETRY

DATE: 14/12/2017 **TIME: 8:30 – 10:30 AM**

Instructions to Candidates

Answer question ONE and any other TWO questions

QUESTION ONE(COMPULSORY) (30 MARKS)

a) Briefly describe in your own words what we study in Algebraic geometry. (3 marks)

b) For each of the following terms give a precise definition.

i. Evaluation homomorphism. (2 marks)

ii. Algebraic variety. (2 marks)

c) Let F be a field

i. State the uniqueness factorization theorem in $F(x)$ (2 marks)

ii. Factorize $f(x) = x^4 + 4$ in to linear factors in $Z_5(x)$ (4 marks)

d) Let $S = \{4x - 5y - 21\} \subset R[x, y]$ describe the algebraic variety $V(S)$ in R^2. (3 marks)

e) Let $f(x) = x^4 + 5x^3 - 3x^2$ and $g(x) = 5x^2 - x + 2$ in $Z_{11}(x)$.

i. Find the product $f(x)g(x)$. (3 marks)

ii. Use the division algorithm to determine the g.c.d of $f(x)$ and $g(x)$. (3 marks)

iii. Describe the kernel of $g(x)$. (3 marks)

f) Show that the set of common zeros in F^n of the polynomials $f_1, f_2, \dots f_n \in F[x]$,where F is a ring is the same as set of common zeros in F^n of all the polynomials in the entire ideal $I =< f_1, f_2, \dots f_n >$. (5 marks)

QUESTION TWO (20 MARKS)

a) Briefly describe each of the following.

i. Fundamental theorem of algebra . (1 mark)

ii. Nullstellensatz for $C[x]$. (2 marks)

iii. Lexicographical order (order lex) for power products. (1 mark)

b) Write the following polynomials in $\mathfrak{R}(x, y, z)$ in decreasing term order using the lexicographical order for power products $x^m y^n z^s$ where $z < y < x$.

i. $3y^2z^5 - 4x + 5y^3z^3 - 8z^7$. (2 marks)

ii. $38 - 4xz + 2yz - 8xy = 3yz^3$. (2 marks)

c) Perform a single step division algorithm reduction that changes the given basis to one having smaller maximum term order using order lex $z < y < x$.

i. $\langle xy + y^3,\ y^3 + z, x - y^4 \rangle$ (2 marks)

ii. $\langle y^2z^3 + 3,\ y^3z^2 - 2z, y^2z^2x + 3 \rangle$ (2 marks)

d) Compute the evaluation homomorphisms:

i) $\emptyset_i(2x^3 - x^2 + 3x + 2)$ (2 marks)

ii) $\emptyset_5(x^3 + 2)(4x^2 + 3)$ (2 marks)

e) By division, reduce the basis $\{ xy^2,\ y^2 - y\}$ for an ideal $I =< xy^2,\ y^2 - y >$ in $R[x, y]$ to one with smaller term size assuming the order lex with $y < x$. (4 marks)

QUESTION THREE (20 MARKS)

a) State the Hilbert basis theorem. Use an example to illustrate. (3 marks)

b) (i) Describe the algebraic variety $V(< x + y - 3z - 8, 2x + y + z + 5 >)$ in R^3 . (5 marks)

(ii) Describe the algebraic variety $V(< f(x), g(x) >)$ in R^3 where,

$f(x) = x^4 + x^3 - 3x^2 - 5x - 2,\quad g(x) = x^3 - 3x^2 - 6x - 8$ (5 marks)

c) Reduce the basis $\{x^2 y - 2,\ xy^2 - y\}$ to obtain a Grobner basis indicating each step. Hence describe the algebraic variety $V(I)$. (7marks)

QUESTION FOUR (20 MARKS)

a) Give the definition of a Grobner basis. (2 marks)

b) Find a Grobner for the following ideal in $\mathfrak{R}[x]$.

i. $\langle x^4 - 4x^3 + 5x^2 - 2x, \; x^3 - x^2 - 4x + 4, \; x^3 - 3x + 2 \rangle$ (3 marks)

c) Find a Grobner for each ideal below in $\mathfrak{R}(x, y)$. Consider order lex with $y < x$. Describe the Algebraic variety in $\mathfrak{R}(x, y)$ where:

i) $\langle x^2y - x - 2, \; xy + 2y - 9 \rangle$ (5 marks)

ii) $\langle x^2y + x + 1, \; xy^2 + y - 1 \rangle$ (5 marks)

d) Let the order of power products in $\mathfrak{R}(w, x, y, z)$ have the order $z < y < x < w$. Find a Grobner basis, for the ideal:

i. $< w - 4x + 3y - z + 2, \; 2w - 2x + y - 2z + 4, \; w - 10x + 8y - z - 5 >$

(3 marks)

e) Identify some importances of Grobner basis in applications. (2 marks)

QUESTION FIVE (20 MARKS)

a) Briefly describe each of the following giving examples.

i. Algebraic element. (4 marks)

ii. Transcendental element. (4 marks)

b) Show that if $f(x) = g(x)q(x) + r(x)$ then the common divisors in $F[x]$ of $f(x)$ and $g(x)$ are the same as the common divisors in $F[x]$ of $g(x)$ and $r(x)$ (6 marks)

c) Let F be a field. Show that if S is nonempty subset of F^n then $I(S) = \{f(x) \in F[x]: f(S) = 0 \; \forall \; s \in S \}$ is an ideal of $F[x]$ (6 marks)