MACHAKOS UNIVERSITY

University Examinations 2016/2017

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR FIRST SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)

SMA 406: FUNCTIONAL ANALYSIS

DATE: 26/7/2017 **TIME: 8:30 – 10:30 AM**

INSTRUCTIONS

Answer question ONE (Compulsory) and any other TWO questions

QUESTION ONE (30 MARKS)

a) Define the following:

i. Metric space (2 marks)

ii. Normed space and Banach space. (2 marks)

iii. Equivalent norms. (2 marks)

b) Show that a subspace Y of a Banach space X is complete if and only if Y is closed in X. (5 marks)

c) Given an operator T: X→ Y defined by Tx=3x .show that T is linear. (3 marks)

d) Show that :

i. $|\|x\| - \|y\|| \leq \|x - y\|$ for all x,y ∈ X . (2 marks)

ii. $|\|x\| - \|y\|| \leq \|x + y\|$ for all x,y ∈ X . (2 marks)

e) Show that f(x) $=\int_a^b x(t)dt$, x ∈ C[a,b] is linear and bounded functional on C[a,b].(6 marks)

f) Show that every convergent sequence in a normed space is a Cauchy sequence. (6 marks)

QUESTION TWO (20 MARKS)

a) Proof the continuous mapping theorem which states that a mapping T on a Metric space X into a metric space Y is continuous if and only if the inverse image of any open subset of Y is an open subset of X. (8 marks)

b) Show that the vector space X=$\mathbb{R}^2$ is a normed space with respect to the following norm

$\|p\| = (x^2 + y^2)^{1/2}$ for all x,y ∈ $\mathbb{R}^2$. (6 marks)

c) Let (x_n) be a weakly convergent sequence in a normed space X ,say $x_n \rightarrow$ x.show that weak limit in x_n is unique . (6 marks)

QUESTION THREE (20 MARKS)

a) Show that the linear space $\mathbb{R}^n$ with $<,>$ defined for arbitrary vectors x=(x_1,x_2,x_3….x_n) , y=(y_1,y_2,y_3……y_n) in $\mathbb{R}^n$ by $< x, y >=\sum_{i=1}^{n} x_i y_i$ is an inner product . (4 marks)

b) Show that an inner product on E defines a norm on E given by $\|x\| =\sqrt{< x, x >}$.(6 marks)

c) If in an inner product space $x_{n \rightarrow} x$ and $y_{n \rightarrow} y$ show that $< x_n, y_n > \rightarrow < x, y >$.

(5 marks)

d) Show that the space l_p with p not equal to zero is not an inner space product space and hence not a Hilbert space . (5 marks)

QUESTION FOUR (20 MARKS)

a) Let X,Y be vector spaces and T:D(T) →R(T) be a linear operator with D(T) ⊂ X and R(T) ⊂ Y .Show that :

i. The inverse of T exist if and only if Tx=0 implies x=0. (5 marks)

ii. If inverse of T exist, it is a linear operator . (5 marks)

b) i Show that the linear space C[0,1] with the function $<,>$ defined for arbitrary f,g ∈ C[0,1] by $< f, g > =\int_{o}^{1} f(x)\overline{g(x)}$ dx is an inner product space . (6 marks)

ii Verify that $< x, 0 >$ =0 for all x ∈X . (4 marks)

QUESTION FIVE (20 MARKS)

a) Show that any closed subset of a compact metric space is compact. (7 marks)

b) Show that any sequentially compact subset of a metric space is bounded and closed.

(7 marks)

c) If (X,d) is a metric space with the discrete metric ,show that the induced topology consists of all the subsets of X. (6 marks)