CYCLE INDICES, SUBDEGREES AND SUBORBITAL GRAPHS OF $PGL(2, q)$ ACTING ON THE COSETS OF ITS SUBGROUPS

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DECLARATION

I wish to declare that this thesis is my original work and has not been presented for a degree award in any other university.

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DEDICATION

I dedicate this thesis to my wife Fridah and my Children Bill, Cindy and Abigail for their support, patience, cooperation and prayer throughout my study period.

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SYMBOLS AND ACRONYMS

S_n	-	The symmetric group of degree n
G_x	-	Stabilizer of a point x in a group G
G_x^{Δ}	-	Transitive constituent of G_x
$\lvert X \rvert$	-	Cardinality of the set X
$\mathbb{Z}$	-	The set of all integers
$\mathbb{Q}$	-	The set of all rational numbers
$\widehat{\mathbb{Q}}$	-	The rational projective line
$PG(1,q)$	-	Projective line
$\mathbb{C}$	-	The set of all complex numbers
$X \times X$		Cartesian product of the set X
$\mathrm{Im}(z)$	-	Imaginary part of z
(x,y)	-	Ordered pair
$\lvert Fix(g) \rvert$	-	Number of points fixed by g
$SL(n,q)$	-	The special linear group
$PGL(n,q)$	-	The projective general linear group
$PSL(n,q)$	-	The projective special linear group
$C_G(g)$	-	Centralizer of g
$GL(n,q)$	-	General linear group

ABSTRACT

The action of $PGL(2,q)$ and $PSL(2,q)$ on the cosets of their subgroups is a very active area in enumerative combinatorics. Most researchers have concentrated on the action of these groups on the cosets of their maximal subgroups. For instance Tchuda computed the subdegrees of the primitive permutation representations of $PSL(2,q)$. Kamuti determined the subdegrees of primitive permutation representations of $PGL(2,q)$. He also constructed suborbital graphs corresponding to the action of $PGL(2,q)$ on the cosets of $D_{2(q-1)}$. However many properties of the action of $PGL(2,q)$ on the cosets of its subgroups are still unknown. This research is mainly set to investigate the action of $PGL(2,q)$ on the cosets some of its subgroups namely; $C_{q-1}, C_{q+1}, P_q, A_4, A_5$ and $D_{2(q-1)}$. Corresponding to each action the disjoint cycle structures, cycle index formulas, ranks and the subdegrees are computed. To obtain cycle index formulas we use a method devised by Kamuti and for the subdegrees and the ranks we use a method proposed by Ivanov *et al*. which uses marks of a permutation group. For the action of $PGL(2,q)$ on the cosets of C_{q-1} the subdegrees are shown to be 1^2 and $(q-1)^{(q+2)}$ and the rank is $q+4$. For P_q the subdegrees are $1^{(q-1)}$ and $q^{(q-1)}$ and the rank is $2(q-1)$. Suborbital graphs for $PGL(2,q)$ acting on the cosets of C_{q-1} are constructed and their properties analysed. We have established that the number of self paired suborbits is $q+2$ and the paired suborbits are 2. Also suborbital graphs corresponding to suborbits whose elements intersect $\{0,\infty\}$ at a singleton have been shown to be of girth 3. Suborbital graph corresponding to the suborbit containing $(0,\infty)$ is found to be of girth 0. Finally suborbital graph corresponding to suborbit with representative of the form $(1,\beta^i)$ is shown to be of girth 4.

CHAPTER ONE

INTRODUCTION

This chapter has five sections. In Section 1.1 we give definitions and preliminary results which will be used throughout the research. Section 1.2 provides the background information of the research we are working on. Section 1.3 gives the statement of the problem. In Section 1.4 we state the objectives of our study. Finally in Section 1.5 the significance of the study is provided.

1.1 Definitions and preliminary results

1.1.1 Definition

Let X be a set. A group G acts on the left of X if for each $g \in G$ and each $x \in X$ there corresponds a unique element $gx \in X$ such that;

i) $(g_1 g_2)x = g_1(g_2 x), \forall g_1, g_2 \in G$ and $x \in X$.

ii) For any $x \in X, Ix = x$, where I is the identity in G .

1.1.2 Definition

Two subgroups H and K of a group G are said to be conjugates if $H = gKg^{-1}$ for some $g \in G$.

1.1.3 Definition

If G is a finite group acting on a finite set X, we define the orbit of $x \in X$ to be;

$$\mathrm{Orb}_G(x) = \{gx \mid g \in G\}.$$

1.1.4 Definition

Let G act on a set X and let $x \in X$. The stabilizer of x in G, denoted by $Stab_G(x)$ is given by, $Stab_G(x) = \{g \in G | gx = x\}$.

This subgroup is also denoted by G_x

1.1.5 Definition

If the action of a group G on a set X has only one orbit, then we say that G acts transitively on X. That is G acts transitively on X if for every pair of points $x, y \in X$, $\exists\ g \in G \ni gx = y$.

1.1.6 Definition

If a finite group G acts on a set X with n elements, each $g \in G$ corresponds to a permutation σ of X, which can be written uniquely as a product of disjoined cycles. If σ has α_1 cycles of length 1, α_2 cycles of length 2, α_3 cycles of length 3,…, α_n cycles of length n; then we say that σ and hence g has a cycle type $(\alpha_1, \alpha_2, ..., \alpha_n)$.

1.1.7 Definition

If a finite group G acts on a set X, $|X| = n$ and $g \in G$ has a cycle type $(\alpha_1, \alpha_2, ..., \alpha_n)$, we define the monomial of g to be,

$mon(g) = t_1^{\alpha_1} t_2^{\alpha_2} ... t_n^{\alpha_n}$, where $t_1, t_2 ..., t_n$ are distinct commuting indeterminates.

1.1.8 Definition

The cycle index of the action of G on X is the polynomial (say over the rational field $\mathbb{Q}$) in $t_1, t_2, ..., t_n$ given by

$$Z(G) = \frac{1}{|G|} \sum_{g \in G} mon(g).$$

1.1.9 Theorem

Let G be a finite transitive permutation group acting on the right cosets of its subgroup H. if $g \epsilon G$ and $|G : H| = n$ then,

$$\frac{\pi(g)}{n} = \frac{|C^g \cap H|}{|C^g|}.$$

where $\pi(g)$ is the number of fixed points and C^g the conjugacy class of $g \in G$.

Proof: For the proof see Kamuti, 1992 p. 5 ■

1.1.10 Lemma

Let g be a permutation with cycle type $(\propto_{1,} \propto_{2\,,} \ldots \ldots \ldots .., \propto_n)$ then,

a) The number $\pi(g^l)$ of 1-cycles in g^l is $\sum_{i|l} i\alpha_i$

b) $\alpha_l = \frac{1}{l} \sum_{i|l} \pi(g^{l/i}) \mu(i)$, where μ is the Mobius function.

Proof: For the proof see Kamuti, 1992 p. 6 ■

1.1.11 Theorem

(Cauchy-Frobenius Lemma)

Let *G* be a finite group acting on a set *X*. Then the number of orbits of *G* is,

$$\frac{I}{|G|}\sum_{g\in G}|Fix(g)|,$$

where $|Fix(g)|$ denotes the number of points in *X* fixed by *g*.

Proof: For the proof see Harary, 1969, p.96 ■

1.1.12 Definition

Let *G* be transitive on *X* and let G_x be the stabilizer of a point $x \in X$. The orbits $\Delta_o = \{x\}, \Delta_1, \Delta_2, ..., \Delta_{r-1}$ of G_x on *X* are known as the suborbits of *G*. The rank of *G* in this case is *r*. The sizes $n_i = |\Delta_i|$ $(i = 0,1,...,r-1)$, often called the 'lengths' of the suborbits, are known as subdegrees of *G*.

1.1.13 Definition

Let Δ be an orbit of G_x on *X*. Define $\Delta^* = \{gx | g \in G, \quad x \in g\Delta\}$, then Δ^* is also an orbit of G_x and is called the G_x-orbit (or the *G*-suborbit paired with Δ). Clearly $|\Delta| = |\Delta^*|$. If $\Delta^* = \Delta$, then Δ is called a self-paired orbit of G_x.

(Wielandt, 1964)

1.1.14 Definition

A permutation representation of a finite group *G* is produced when a group acts on a finite set $X = \{a_1, a_2, \dots\dots\dots\dots\dots\dots, a_n\}$, where *n* is the cardinality of *X*. A permutation representation, P_G is the set of permutation P_g on X, each of which is associated with an element $g \in G$ so that P_G and G are homomorphic;

$$P_g P_{g'} = P_{gg'}, \text{ for every } g, g' \in G.$$

We say that two permutation representations of G on X_1 and X_2 are equivalent if there exists a bijection $\varphi: X_1 \to X_2$ such that $\varphi(g(x_1) = g\varphi(x_1)$ for all $g \in G$ and all $x_1 \in X_1$.

Let *H* be a subgroup of index *n* in *G*. The set of left cosets of *H* in *G* partitions *G* i.e ;

$G = g_1H \cup g_2H \cup \ldots\ldots\ldots\ldots\ldots \cup g_nH$, where $g_1 = I$ and $g_i \in G$. Consider the set of the left cosets $\{g_1H, g_2H, \ldots\ldots\ldots\ldots\ldots, g_nH\}$. For any $g \in G$, the set of permutations of degree *n*;

$$G\,(/H)(g) = \begin{Bmatrix} g_1H \ldots\ldots g_nH \\ gg_1H \ldots gg_{nH} \end{Bmatrix},$$

constructs a permutation representation of G, which is sometimes called the coset representation of *G* by *H* and we shall denote it by $G(/H)$. The degree of $G(/H)$ is $|G|/|H| = n$ and it is a transitive permutation representation.

We now state two important results which have been proved in Burnside (1911) pages 236-238.

1.1.15 Theorem

Suppose that the number of conjugacy classes of subgroups in a finite group *G* is *s* (where a set of conjugacy class is counted once). If we collect a complete set;

$G_1, G_2, \ldots \ldots \ldots, G_s$ in ascending order of their sizes i.e $|G_1| \leq |G_2| \leq \cdots \ldots \ldots \leq |G_s|$, where;

$G_1 =$ identity and $G_s = G$, then the set of corresponding coset representation ; $G(/G_i), (i = 1, 2, \ldots\ldots\ldots\ldots, s)$ is the complete set of different transitive permutation representation of G.

1.1.16 Theorem

Any permutation representation P_G of a finite group G acting on X can be reduced into transitive coset representations with the following equation:

$$P_G = \sum \alpha_i G(/G_i), (i = 1, 2, \ldots\ldots\ldots\ldots, s),$$

where the multiplicity α_i is a non-negative integer.

1.1.17 Definition

i) Burnside's definition of marks

Let P_G be a permutation representation (transitive or intransitive) of G on X. The mark of the subgroup H of G in P_G is the number of points of X fixed by every permutation of H. incase $G(/G_i)$ is a coset representation ; $m(G_j, G_i, G)$, the mark of G_j in $G(/G_i)$ is the number of cosets of G_i in G left fixed by every permutation of G_j.

ii) White's definition of marks;

$$m(G_j, G_i, G) = \frac{1}{|G_i|} \sum_{g \in G} x(g^{-1} G_j g \subseteq G_i),$$

where x (statement) $= \begin{cases} 1, & \textit{if the statement is true} \\ 0, & \textit{if the statement is untrue} \end{cases}$

iii) Ivanov *et al*. definition of marks

Ivanov *et al*. (1983) defined the mark in terms of normalizers of subgroups of a group as; if $G_j \leq G_i \leq G$ and $(G_{j1}, G_{j2}, \dots \dots \dots .., G_{jn})$ is a complete set of conjugacy class representatives of subgroups of G_i that are conjugate to G_j in G, then;

$$m(G_j, G_i, G) = \sum_{k=1}^{n} \left| N_G(G_{j_k}) : N_{Gi}(G_{j_k}) \right|.$$

In particular when $n = 1$, G_j is conjugate in G_i to all subgroups $G_{j'}$ that are contained in G_i and are conjugate to G_j in G and;

$$m(G_j, G_i, G) = \left| N_G(G_j) : N_{G_i}(G_j) \right|.$$

It can be shown that these definitions are equivalent. (Kamuti,1992 p. 77)

1.1.18 Definition

The table of marks of a group G is the matrix $M(G)$, with

(i, j) entry m_{ij} equal to $m(G_j, G_i, G)$, the mark of the subgroup G_j in the coset representation $G(/G_i)$.

i.e;

Table 1.1.1: Table of marks of a group *G*

	G_1	G_2 ………………	G_s
$G(/G_1)$	m_{11}	m_{12} …………	m_{1s}
$G(/G_2)$	m_{21}	m_{22} …………	m_{2s}
….	…	…. ……….	…
….	…	…. ……….	….
$G(/G_s)$	m_{s1}	m_{s2} …………	m_{ss}

It can be shown that $m_{ij} = 0$ unless G_j is conjugate to a subgroup of G_i and that $m_{ii} \geq 1$ for any i , because of this, if representative subgroups are numbered in increasing order of size, the table of marks is lower triangular and since it has no zero entries on the main diagonal, it is an invertible matrix. (Kamuti, 1992, p. 78)

By Theorem 1.1.16 the multiplicities $\propto_i$ are obtained by using the table of marks as;

$$\omega_j = \sum_{i=1}^{s} \alpha_i \, m_{ij}, \quad (j = 1, 2, \dots \dots, s),$$

where ω_j is the mark of G_j in P_G. If $\omega = (\omega_1, \omega_2, \dots \dots, \omega_s)$ is a vector with components ω_j, the marks of G_j;

$(j = 1,2, \dots, s)$ in the permutation representation P_G of *G* on *X*,

$\alpha = (\alpha_1, \alpha_2 \dots \dots \dots, \alpha_s)$ is a vector with the components the multiplicities α_i in Theorem 1.1.16 and *M(G)* is the table of marks of G, then

$$\omega = \alpha M(G).$$

If we denote by Q_i the number of suborbits Δ_j on which the action of H is equivalent to its action on the cosets of H_i $(i = 1, 2, \dots \dots, t)$, by computing all the Q_i we get the subdegrees of (G, X). Hence we have;

1.1.19 Theorem

The number Q_i satisfy the system of equations;

$$\sum_{i=j}^{t} Q_i m(H_j, H_i, H) = m(H_j, H, G)$$

for each $j = 1,2, \dots \dots, t$.

Proof: For proof see Kamuti, 1992, p.78 ■

1.1.20 Definition

A graph is a diagram consisting of a set *V* whose elements are called vertices, nodes or points and a set *E* of unordered pair of vertices called edges or lines. We denote such a graph by $G(V,E)$. Two vertices *u* and *v* of a graph $G(V,E)$ are said to be adjacent if there is an edge joining them. This is denoted by $\{u,v\}$ and sometimes by *uv*. In this case *u* and *v* are said to be incident to such edge.

A graph consisting of one vertex and no edge is called a trivial graph.

If we allow existence of loops (edges joining vertices to themselves) and multiple edges (more than one edge joining two distinct vertices), then we get a multigraph.

A graph with no loops or multiple edges is called a simple graph.

The degree (valency) of a vertex *v* of $G(V,E)$ is the number of edges incident to *v*

1.1.21: Definition

For any positive integer n define $\mu(n)$ (mobius function of n) as the sum of the primitive n[th] root of unity. It has values $\{-1, 0, 1\}$ depending on the factorization of n into prime factors:

$\mu(n) = 1$ if n is a square free positive integer with an even number of prime factors;

$\mu(n) = -1$ if n is a square free positive integer with an odd number of prime factors;

$\mu(n) = 0$ if n has squared prime factors.

1.2 Background information

1.2.1 Projective General Linear Group

The Projective General Linear Group $PGL(2, q)$ over a finite field $GF(q)$, where $q = p^f$, p a prime number and f a natural number, is a group consisting of all linear fractional transformations of the form;

$$x \to \frac{ax + b}{cx + d} \ ,$$

where $x \in PG(1, q) = GF(q) \cup \{\infty\}$, the projective line, $a, b, c, d \in GF(q)$ and $ad - bc \neq 0$.

It is the factor group of the general linear group by its centre. That is;

$$PGL(2, q) = \frac{GL(2, q)}{Z(GL(2, q))} .$$

Thus a transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in GF(q), ad - bc \neq 0$ and $\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$

$k \in GF(q)^*$ are taken to be the same in $PGL(2,q)$, where $GF(q)^*$ are non zero elements of $GF(q)$.

The order of the projective general linear group is,

$$\frac{|GL(2,q)|}{|Z(GL(2,q))|} = \frac{(q^2-1)(q^2-q)}{q-1} = q(q^2-1).$$

$PSL(2,q)$ is a subgroup of $PGL(2,q)$ with $ad - bc = 1$.

It is simple for $q > 3$. It is also 2-transitive on the $PG(1,q)$ of degree $q+1$ and it is of order,

$$|PSL(2,q)| = \frac{q(q^2-1)}{k},$$

where $k = (q-1, 2)$.

If q is a power of 2, then $PGL(2,q) \cong PSL(2,q)$.

(Dickson, 1901)

1.2.2 Properties of $PGL(2,q)$ acting on the projective line

Non identity transformations act non- trivially on the projective line.

$PGL(2,q)$ acts doubly transitive on the projective line.

$PGL(2,q)$ also acts sharply 3-transitively on the projective line.

(Huppert 1967)

1.2.3 Subgroups of $PGL(2,q)$

$PGL(2,q)$ is the union of three conjugacy classes of subgroups each of which intersects with each other at the identity subgroup, and these are;

a) Commutative subgroups of order q

$PGL(2,q)$ has commutative subgroups of order q. We denote it by P_q.

Non- identity transformations in P_q have only ∞ as a fixed point in $PG(1,q)$. It is of the form;

$$P_q = \left\{\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} | b \in GF(q)\right\},$$

The normalizer of P_q in $PGL(2,q)$ has transformations of the form;

$$S = \begin{pmatrix}c & e\\ 0 & d\end{pmatrix} \text{ where } e \in GF(q) \text{ and } c,d \in GF(q)^*.$$

Thus ,

$$\left|N_{PGL(2,q)}(P_q)\right| = q(q-1).$$

Therefore the number of subgroups of $PGL(2,q)$ conjugate to P_q is $q+1$. These $q+1$ subgroups have no transformation in common except the identity. All the conjugate subgroups of order q contain q^2-1 distinct non identity transformations of order p.

b) Cyclic subgroups of order $q-1$

$PGL(2,q)$ also contains a cyclic subgroup C_{q-1} of order $q-1$. Each of its non identity transformations fixes 0 and ∞. It is of the form;

$$C_{q-1} = \left\{\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix} | a,b \in GF(q)^*\right\}.$$

The normalizer of C_{q-1} is generated by the transformations T and W

Where; $$T = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix},$$

and h is a primitive element in $GF(q)$ and

$$W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus ,

$\left|N_{PGL(2,q)}(C_{q-1})\right| = 2(q-1).$

Therefore the number of subgroups of $PGL(2,q)$ conjugate to C_{q-1} is $\frac{1}{2}q(q+1)$.

These conjugate cyclic subgroups have no transformation in common except the identity and thus contain $\frac{1}{2}q(q+1)(q-2)$ non identity transformations

c) Cyclic subgroups of order $q+1$

Another subgroup of $PGL(2,q)$ is the cyclic subgroup C_{q+1} of order $q+1$. This consists of elliptic transformations. All the non identity elements of the cyclic subgroup of order $q+1$ fixes no element. The normalizer of this subgroup is a dihedral subgroup of order $2(q+1)$.

Therefore the number of subgroups of $PGL(2,q)$ conjugate to this cyclic subgroup of order $q+1$ is $\frac{1}{2}q(q-1)$. Since the $\frac{1}{2}q(q-1)$ conjugate subgroups have only identity in common so, they contain $\frac{1}{2}q^2(q-1)$ non identity transformations.

A simple enumeration shows that all the transformations of the commutative subgroups of order q, cyclic subgroups of order $(q-1)$ and cyclic subgroups of order $(q+1)$ exhaust all the transformations of $PGL(2,q)$. That is;

$$1 + (q^2 - 1) + \left(\frac{1}{2}q(q+1)(q-2)\right) + \left(\frac{1}{2}q^2(q-1)\right) = q(q^2 - 1),$$

which is the order of $PGL(2,q)$. (Dickson, 1901 p. 262)

1.2.3.1 Theorem

a) Let $\wp$ be the following set of subgroups of $G = PGL(2,q)$;

$$\wp = \{P_q^g, C_{q-1}^g, C_{q+1}^g | g \in G\}.$$

Then each non-identity elements of G is contained in exactly one group in $\wp$. (Thus the set $\wp$ form a partition of G.)

b) Let $\pi(g)$ be the number of fixed points of $g \in G$ on the $PG(1,q)$. If we define

$$\tau_i = \{g | g \in G, \pi(g) = i\};$$

then

$\tau_0 = \cup\, (c_{q+1} - I)^g, \tau_1 = \cup\, (P_q - I)^g, \tau_2 = \cup\, (c_{q-1} - I)^g.$

(Huppert, 1967 p.193)

NB: Those permutations with precisely one fixed point on the $PG(1,q)$ are the parabolic elements. Those with two fixed points are the hyperbolic elements and those with no fixed points are the elliptic elements.

1.2.3.2 Lemma

If g is elliptic or hyperbolic of order greater than 2 or if g is parabolic, then the centralizer in $PGL(2,q)$ consists of all elliptic (respectively, hyperbolic, parabolic) elements with the same points set, together with the identity elements. On the other hand if g is elliptic or hyperbolic of order 2, then its centralizer is the dihedral group of order $2(q+1)$ or $2(q-1)$ respectively. (Dickson, 1901)

1.3 Statement of the problem

Although the action of $PGL(2,q)$ on cosets of its subgroups has been known for many years little has been done in this area especially the action on the non maximal subgroups. A lot of attention has been given to the action of $PGL(2,q)$ and $PSL(,q)$ on their maximal subgroups. This study concentrates more on the action of $PGL(2,q)$ on the cosets of some of its non maximal subgroups namely; $C_{q-1}, C_{q+1}, P_q, A_4, A_5$ and maximal subgroup $D_{2(q-1)}$. Corresponding to each action the disjoint cycle structures, cycle index formulas, ranks and the subdegrees are computed. Suborbital graphs corresponding to the action $PGL(2,q)$ on the cosets of C_{q-1} is constructed and their theoretic properties such as, self pairing, pairing and the girths sizes is investigated.

1.4 Objectives

1.4.1 General objective

To investigate the actions of $PGL(2,q)$ on the cosets of some of its subgroups namely; $C_{q-1}, C_{q+1}, P_q, A_4, A_5$ and $D_{2(q-1)}$.

1.4.2 Specific objectives

i) To find the disjoint cycle structures of elements of $PGL(2,q)$ in each of the corresponding action.

ii) To find the cycle index formulas of $PGL(2,q)$ corresponding to each action.

iii) To determine the ranks and subdegrees of $PGL(2,q)$ corresponding to these actions.

iv) To construct the suborbital graphs corresponding to the action of $PGL(2,q)$ on the cosets of C_{q-1} and to investigate their theoretic properties.

1.5 Significance of the Study

This research obtained new properties and generalized the existing results. In addition; the results of this study provides valuable information to the graph theorists. Graphs have several practical applications in real life situation as well as in other fields of study. For instance, they can be used to determine the shortest or longest distance between places on the earth's surface. In Chemistry, graph theory can be used to study the structure of molecules as it makes a natural model for molecules, where vertices represent atoms and edges bonds. The cycle index formula can be used to find the counting series for unlabelled graphs which can be used to find the number of isomers of a given molecule.

CHAPTER TWO

LITERATURE REVIEW

This chapter considers the work which has been done by other researchers. It has three sections. In Section 2.1 we provide the literature review of cycle indices. Section 2.2 gives a review of ranks and the subdegrees and finally Section 2.3 provides what has been done on the suborbital graphs.

2.1 Cycle index

The cycle index of a permutation group was introduced by Redfield in 1927, but he called it the group reduction function (Grf). He studied some of the links between combinatorial analysis and permutation groups.

Polya independently rediscovered the same function in 1937. He used it to count graphs and chemical compounds via the famous Polya Enumeration Theorem. Through this theorem, the cycle index becomes a very powerful tool in enumeration. After Polya several authors have used the cycle index in enumeration Clerke (1990) proved the identity concerning the cycle index polynomial of the symmetric group and presented its consequences.

Herald (1996) computed the Polya cycle indices for the natural actions of the general linear groups and the affine groups (on vector space) and for the projective linear groups (on a projective space) over a finite field. He also demonstrated how to enumerate isometric classes of linear codes by using the cycle indices. Kamuti and Obong'o (2002) derived the cycle index formula of $S_n^{[3]}$. Kamuti and Njuguna (2004) derived the cycle index formula of the reduced ordered r-group.

Kamuti (2004) showed how the cycle index of a semidirect product of a group $G = M \rtimes H$ can be expressed in terms of cycle indices of M and H by considering special types of semidirect products called the Frobenius groups.

Kamuti (2012) extended the work of Kamuti (2004). He expressed the cycle index of $G = M \times H$ (internal direct product) in terms of the cycle indices of M and H when G acts on the cosets of H in G.

Mogbonju *et al.* (2014) found the cycle index of permutation groups especially the symmetric groups, the alternating group and the number of orbits of A_n.

2.2 Ranks and Subdegrees

Wielandt (1955) proved that a primitive group of degree 2*p*, *p* a prime, has rank of at most 3. Higman (1964) showed that any 4-fold transitive group has rank 3 when considered as a group of permutations of the unordered pairs from distinct points. Higman (1970) calculated the rank and the subdegrees of the symmetric group S_n acting on 2-element subsets from the set $X = \{1, 2, \ldots, n\}$. He showed that the rank is 3 and the subdegrees are $1, 2(n-2), \binom{n-2}{2}$.Quirin (1971) studied primitive permutation groups with small suborbits. He was able to classify all primitive permutation groups *G* which have a suborbit Δ of length 4 for which $G_x^{\Delta} \cong A_4$ or $G_x^{\Delta} \cong S_4$ is faithful.

Cameroon (1975) studied suborbits in transitive permutation groups. He proved that if *G* is primitive on *X* and G_x is doubly transitive on all non-trivial suborbits except possibly one, with $|G_x| > 2$, then *G* has rank at most 4.

Neumann (1977) extended the work of Higman to finite permutation groups, edge coloured graphs and matrices. In this paper, he drew the Petersen (1898) graph as a suborbital graph corresponding to one of the non-trivial suborbits of S_5 on unordered pairs from {1,2,3,4,5}.

Numata (1978) studied primitive rank 5 permutation groups. He proved that if *G* is a primitive rank 5 permutation group on a finite set *X*, and the stabilizer G_x of a point $x \in X$ is doubly transitive on $\Delta_1(x)$ and $\Delta_2(x)$, where $\Delta_1(x)$ and $\Delta_2(x)$ are two G_x-orbits with $\Delta_1 \circ \Delta_1^* \neq \Delta_2 \circ \Delta_2^*$, then *G* is isomorphic to the small Janko simple group and $|X| = 266$.

Ivanov *et al.* (1983) gave a method of computing subdegrees of transitive permutation group using the table of marks. They gave a sporadic simple group J_1 as an example.

Tchuda (1986) computed the subdegrees of the primitive permutation representations of $PSL(2,q)$. Cai *et al.* (2004) extended the work of Quirin (1971) on primitive permutation groups with small suborbits. They came up with a precise list of primitive permutation groups with a suborbit of length 4. In particular they showed that there exists no examples of such groups with the point stabilizer of order $2^4 3^6$, clarifying an uncertain question (since 1970s).

Kamuti (2006) computed the subdegrees of primitive permutation representations of $PGL(2,q)$ using a method proposed by Ivanov *et al*. (1983) which uses marks of a permutation group. For instance, in the action of $PGL(2,q)$ on the cosets of S_4, he found that the rank is,

$r=\frac{q^3+189q-82}{576}$. Nyaga *et al.* (2011) computed the ranks and the subdegrees of the symmetric group S_n acting on unordered $r-$element subsets. They proved that the rank is $r+1$ if $n \geq 2r$. They also showed that the subdegrees are;

$$1, r\binom{n-r}{r-1}, \binom{r}{2}\binom{n-r}{r-2}, \binom{r}{3}\binom{n-r}{r-3} \dots \dots \dots . \binom{r}{r-1}\binom{n-r}{1}, \binom{n-r}{r}.$$

2.3 Suborbital graphs

Sims (1967) introduced suborbital graphs corresponding to the nontrivial suborbits of a group *G* acting on *X*. He defined a suborbital graph Γ_i corresponding to suborbital $O_i \subseteq X \times X$ as a graph in which the vertex set is *X* and the edge set *E* consists of directed edges xy such that $(x, y) \in O_i$. After Sims several researchers have studied suborbital graphs.

Jones *et al*. (1991) investigated the action of $PSL(2, \mathbb{Z})$ on the rational projective line. They observed that the action on $\widehat{\mathbb{Q}}$ was transitive but imprimitive.

They also constructed the suborbital graphs corresponding to the above action and the simplest was the Farey graph F. They found that F is connected and contains undirected triangles.

Kamuti (1992) devised a method for constructing some of the suborbital graphs of $PGL(2, q)$ acting on the cosets of its maximal dihedral subgroup of order $2(q-1)$.

This method gave an alternative way of constructing the Coxeter graph which was first constructed by Coxeter (1983). This is a non-Hamiltonian cubic graph on 28 vertices and 42 edges with girth 7.

Akbas (2001) extended the work of Jones *et al.* (1991) on the action of the modular group on the rational projective line $\widehat{\mathbb{Q}}$. He was able to prove their conjecture, that a suborbital graph for a modular group is a forest if and only if it contains no triangles.

Refik (2005) characterized all circuits in the suborbital graph for the normalizer of $\Gamma_0(\mathrm{m})$ when *m* is a square- free positive integer.

Refik (2009) dealt with the conjuncture given in Refik (2005) that when the normalizer of $\Gamma_0(\mathrm{N})$ acts transitively on $\mathbb{Q} \cup \{\infty\}$, any circuit in the suborbital graph $G(\infty, \frac{u}{n})$ for the normalizer of $\Gamma_0(\mathrm{N})$, is of the form;

$$v \to T(v) \to T^2(v) \to \cdots \to T^{k-1}(v) \to v,$$

where $n > 1, v \in \mathbb{Q} \cup \{\infty\}$ and T is elliptic mapping of order *k* in the normalizer of $\Gamma_0(\mathrm{N})$.

Kader *et al.* (2010) found the number of sides of circuits in suborbital graph for the normalizer of $\Gamma_0(\mathrm{m})$ in $PSL(2, \mathbb{R})$, where *m* is of the form $2p^2$, p a prime and $p \equiv 1(mod4)$. Also they gave a theoretical result which says that the prime divisors p of $2u^2 \pm 2u + 1$ are of the form $p \equiv 1(mod4)$. Bahadir *et al*. (2010) examined $\Gamma_0(\mathrm{N})$-orbits on $\mathbb{Q}$ and the suborbital graphs for $\Gamma_0(\mathrm{N})$. They showed that each such suborbital graph is a disjoint union of subgraphs whose vertices form blocks of imprimitivity for $\Gamma_0(\mathrm{N})$. They also proved that the subgraphs are vertex $\Gamma_0(\mathrm{N})$ $-$transitive and edge $\Gamma_0(\mathrm{N})$ $-$transitive. Serkan *and* Bahadir . (2011) showed that the conjugate elliptic elements of the modular group Γ and of congruence subgroup $\Gamma_0(p)$ give rise to conjugate circuit corresponding to the related elliptic elements in the Farey graphs F and in the suborbital graph $F_{u,p}$ of the action of $\Gamma_0(p)$, respectively.

Murat *et al.* (2012) investigated suborbital graphs for the action of the normalizer of $\Gamma_0(\mathrm{N})$ on $PSL(2,\mathbb{R})$, where N is of the form 2^8p^2, $p > 3$ and p is a prime number. In addition they gave the condition to be a forest for normalizer in suborbital graph $F(\infty, \frac{u}{2^8p^2})$.

Kamuti *et al*. (2012) investigated some properties of the action of Γ'_∞ (the stabilizer of ∞ in Γ' (the modular group)) acting on the set of $\mathbb{Z}$. They showed that the action is simply transitive and imprimitive. They also examined the properties of the suborbital graphs corresponding to this action.

Thus the cycle index formula, ranks, subdegrees, and the suborbital graphs of $PGL(2,q)$ acting on the cosets of most of its subgroups have not been published so far. This is the area we mainly concentrated on our research.

CHAPTER THREE

PERMUTATION REPRESENTATIONS OF $G = PGL(2, q)$

In this chapter we shall determine the disjoint cycle structure of elements of G acting on the cosets of its subgroups. This chapter is divided into six sections. In Section 3.1 to Section 3.6 we determine the disjoint cycle structure of elements of G acting on the cosets of $C_{q-1}, C_{q+1}, P_q, A_4, A_5$ and $D_{2(q-1)}$ respectivley. Though Kamuti (1992) had already done this for the cosets of $D_{2(q-1)}$ using the concept of pair group action, we will also work on this subgroup using geometric arguments (i.e Theorem 1.1.9 and Lemma 1.1.10 (b)).

3.1 Representation of *G* on the cosets of $H = C_{q-1}$

From Section 1.2.3, G contains a cyclic subgroup H of order $q-1$ whose every non-identity elements fixes two elements. If g is an element in G, we may want to find the disjoint cycle structures of the permutation g' induced by g on the cosets of H.Our computation will be carried out by each time taking an element g of order d in G from $\tau_{1,}\tau_2$ and τ_0 respectively.

To find the disjoint cycle structures of $g \in G$ we use Lemma 1.1.10(b) and thus we need to determine $|C^g|$, $|C^g \cap H|$ and $\pi(g)$ using Theorem 1.1.9. We easily obtain $|C^g|$ using Lemma 1.2.3.2, but we need to distinguish between $d = 2$ and $d > 2$. So if $d > 2$ then we have;

$$|C^g| = \frac{|G|}{|C(g)|},$$

where $|C(g)|$ is the order of centralizer of g.

So if $g \in \tau_0$, then

$$|C^g| = \frac{q(q^2-1)}{(q+1)} = q(q-1).$$

Also if $g \in \tau_1$, then

$$|C^g| = \frac{q(q^2-1)}{q} = q^2 - 1.$$

Finally if $g \in \tau_2$, then

$$|C^g| = \frac{q(q^2-1)}{(q-1)} = q(q+1).$$

If $d = 2$ then;

For $g \in \tau_0$, we have;

$$|C^g| = \frac{q(q^2-1)}{2(q+1)} = \frac{q(q-1)}{2}.$$

Again if $g \in \tau_1$, then

$$|C^g| = \frac{q(q^2-1)}{q} = q^2 - 1.$$

Finally if $g \in \tau_2$, then

$$|C^g| = \frac{q(q^2-1)}{2(q-1)} = \frac{q(q+1)}{2}.$$

After finding $|C^g|$, then we need to obtain $|C^g \cap H|$. If no $h \in H$ with $|h| = d$ (order of g) exists then $|C^g \cap H| = 0$ and if such an h exists then;

$$|C^g \cap H| \neq 0.$$

If $g \in \tau_1$ or $g \in \tau_0$ then;

$$|C^g \cap H| = 0.$$

If $g \in \tau_2$ and $d = 2$, then $\varphi(2) = 1$. H has a single element of order 2. Each subgroup of G conjugate to H has one element of order 2. So

$$|C^g \cap H| = 1,$$

if $g \epsilon \tau_2$ and $d = 2$.

If $g \in \tau_2$ and $d > 2$, then

$$|C^g \cap H| = 2.$$

Now applying Theorem 1.1.9, if $g \in \tau_1$ and $g \in \tau_0$ we have,

$$\pi(g) = 0.$$

This is true for all $d = 2$ and $d \neq 2$.

If $g \in \tau_2$ and $d \neq 2$,

$$\pi(g) = 2.$$

For $g \epsilon \tau_2$ and $d = 2$,

$$\pi(g) = 2.$$

The values of $\pi(g)$are displayed in Table 3.1.1 below.

Table 3.1.1: No. of fixed points of elements of G acting on the cosets of $\boldsymbol{C_{q-1}}$

Case		$\|C^g\|$	$\|C^g \cap H\|$	$\pi(g)$
I	$g \in \tau_0\ d > 2$	$q(q-1)$	0	0
	$d = 2$	$\frac{q(q-1)}{2}$	0	0
II	$g \in \tau_1\ d > 2$	$q^2 - 1$	0	0
	$d = 2$	$q^2 - 1$	0	0
III	$g \in \tau_2\ d > 2$	$q(q+1)$	2	2
	$d = 2$	$\frac{q(q+1)}{2}$	1	2

After obtaining $\pi(g)$, we now proceed to calculate in details the disjoint cycles structures of elements g' in this representation using Lemma 1.1.10(b).

Case I

$g \in \tau_0$

If $1 \leq l < d$, we deduce from Table 3.1.1 that $\pi(g^l) = 0$ and hence by Lemma 1.1.10(b) $\alpha_l = 0$.

When $l = d$, then we have;

$$\alpha_d = \frac{1}{d}\pi(g^d)\mu(1)$$

$$= \frac{q(q+1)}{d}.$$

Case II

$g \in \tau_1$

Using Lemma 1.1.10 (b) and Table 3.1.1, if $1 \leq l < p$, $\pi(g^l) = 0$ and thus $\alpha_l = 0$. If $l = d = p$, then

$$\alpha_p = \frac{1}{p}\pi(g^{p/1})\mu(1)$$

$$= \frac{1}{p}\pi(g^p)\mu(1)$$

$$= p^{f-1}(q+1).$$

Case III

$g \in \tau_2$

From Table 3.1.1 $\pi(g) = 2$ for $1 \leq l < d$. Hence for $1 < l < d$, $\alpha_l = 0$. If $l = d$, then

$$\alpha_d = \frac{1}{d}\left[\pi(g^{d/1})\mu(1) + 2(\sum_{i|d}\mu(i) - \mu(1))\right]$$

$$= \frac{1}{d}\left[q(q+1) - 2\right]$$

$$= \frac{1}{d}\left[(q-1)(q+2)\right].$$

Also if $l = 1$, then $\pi(g^1) = 2$ and thus,

$$\alpha_1 = \left[\pi(g^{1/1})\mu(1)\right]$$

$$= 1\left[2\right] = 2.$$

Summary of results

Table 3.1.2: Disjoint cycle structures of elements of G on the cosets of $\boldsymbol{C_{q-1}}$

	τ_1		τ_0		τ_2	
Cycle length of g'	1	p	1	d	1	d
No. of cycles	0	$p^{f-1}(q+1)$	0	$\frac{q(q+1)}{d}$	2	$\frac{(q-1)(q+2)}{d}$

3.2 Representation of *G* on the cosets of $\boldsymbol{H = C_{q+1}}$

To compute the disjoint cycle structures of g' we first need to determine $\pi(g)$ by using Theorem 1.1.9 and find α_l by applying Lemma 1.1.10(b). Before we obtain $\pi(g)$ we first need to determine $|C^g|$ and $|C^g \cap H|$. Since $|C^g|$ is the same as in Section 3.1, so we only need to find $|C^g \cap H|$. If $g\epsilon\tau_1$ and τ_2 then,

$$|C^g \cap H| = 0.$$

. If $g\epsilon\tau_0$ then,

$$|C^g \cap H| = \begin{cases} 1 \; if \; d = 2 \\ 2 \; if \; d \neq 2 \end{cases}.$$

This is because if $d = 2$ $\varphi(2) = 1$. So it has a single element of order 2. Each subgroup of G conjugate to H has one element of order 2.

If $g\epsilon\tau_0$ and $d \neq 2$ we have;

$$|C^g \cap H| = 2$$

We now proceed to obtain $\pi(g)$ using Theorem 1.1.9. For $g\epsilon\tau_1$ and $g\epsilon\tau_2$ we have;

$$\pi(g) = 0.$$

For $g\epsilon\tau_0$ and $d \neq 2$,

$$\pi(g) = 2.$$

For $g\epsilon\tau_0$ and $d = 2$,

$$\pi(g) = 2.$$

The values of $\pi(g)$ are displayed in Table 3.2.1 below

Table 3.2.1: No. of fixed points of elements of *G* acting on the cosets of $\boldsymbol{C_{q+1}}$

Case		$\lvert C^g \rvert$	$\lvert C^g \cap H \rvert$	$\pi(g)$
I	$g \in \tau_0$ $d > 2$	$q(q-1)$	2	2
	$d = 2$	$\frac{q(q-1)}{2}$	1	2
II	$g \in \tau_1$ $d > 2$	$q^2 - 1$	0	0
	$d = 2$	$q^2 - 1$	0	0
III	$g \in \tau_2$ $d > 2$	$q(q+1)$	0	0
	$d = 2$	$\frac{q(q+1)}{2}$	0	0

After obtaining $\pi(g)$ we use the same approach as in Section 3.1 above to find the disjoint cycle structures of elements of *G* on the cosets of C_{q+1}.Therefore the results is as shown in Table 3.2.2 below.

Summary of results

Table 3.2.2: Disjoint Cycle structures of elements of G on the cosets of $\boldsymbol{C_{q+1}}$

	τ_1		τ_0		τ_2	
Cycle length of g'	1	p	1	d	1	d
No. of cycles	0	$p^{f-1}(q-1)$	2	$\frac{(q+1)(q-2)}{d}$	0	$\frac{q(q-1)}{d}$

3.3 Representations of G on the cosets of $H = P_q$

To obtain the disjoint cycle structures of g we use Theorem 1.1.9 to get $\pi(g)$. We compute α_l using Lemma 1.1.10(b). To find $\pi(g)$ we first need to obtain $|C^g|$ and $|H \cap C^g|$. Since $|C^g|$ is the same as in Section 3.1, so we only need to determine $|H \cap C^g|$.

If $g \in \tau_0$ and $g \in \tau_2$,

$$|H \cap C^g| = 0.$$

If $g \in \tau_1$ then;

$$|H \cap C^g| = q - 1,$$

Now using Theorem 1.1.9 and for $g \in \tau_0$ or $g \in \tau_2$, we have;

$$\pi(g) = 0.$$

For τ_1 ,

$$\pi(g) = q - 1.$$

The values of $\boldsymbol{\pi(g)}$ are displayed in Table 3.3.1 below;

Table 3.3.1: No. of fixed points of elements of G on the cosets of $\boldsymbol{p_q}$

Case		$\lvert C^g \rvert$	$\lvert C^g \cap H \rvert$	$\pi(g)$
I	$g \in \tau_0$	$q(q-1)$	0	0
II	$g \in \tau_1$	q^2-1	$q-1$	$q-1$
III	$g \in \tau_2$	$q(q+1)$	0	0

In this section we will only discus case II and use similar approach to find the disjoint cycle structures of elements of G on the cosets of P_q for the other two cases.

$g \in \tau_1$

From the results in Table 3.3.1 $\pi(g^l) = q-1$ for $1 \le l < d$. Hence $\alpha_l = 0$ for $1 < l < d$. If $l = d = p$, then;

$$\alpha_p = \frac{1}{p}\left[\pi(g^p)\mu(1) + (q-1)\left(\sum_{1 \neq i | l} \mu(i) - \mu(1)\right)\right]$$

$$= \frac{1}{p}[(q^2-1) - (q-1)]$$

$$= \frac{q}{p}(q-1).$$

If $l = 1$, then we have,

$$\alpha_1 = \pi(g^1)\mu(1)$$

$$= q-1.$$

Therefore the results are as shown in Table 3.3.2 below.

Summary of results

Table 3.3.2: Disjoint Cycle structures of elements of G on the cosets of $\boldsymbol{p_q}$

	τ_1		τ_0		τ_2	
Cycle length of g'	1	p	1	d	1	d
No. of cycles	$q-1$	$\frac{q(q-1)}{p}$	0	$\frac{(q^2-1)}{d}$	0	$\frac{(q^2-1)}{d}$

3.4 Representations of *G* on the cosets of $\boldsymbol{H = A_4}$

G contains a subgroup *H* isomorphic to A_4 if and only if $p > 2$ or $p = 2$ and

$f \equiv 0 \; mod \; 2$.

H contains 3 conjugate elements of order 2 and 8 elements of order 3 which lie in two mutually inverse conjugacy classes of 4 elements. This can be illustrated in Table 3.4.1 below.

Table 3.4.1: Cycle structure of elements of $\boldsymbol{A_4}$

Cycle structures	No. of permutation	order
I	1	1
(ab)(cd)	3	2
(abc)	8	3

If $p = 2$ there is a single conjucacy class of elements of order 2 in *G* containing $q^2 - 1$ elements. If $p \neq 2$, G contains 2 conjugacy classes of elements of order 2 consisting of $\frac{q(q+1)}{2}$ elements of order two and the other $\frac{q(q-1)}{2}$.

From the information given above, we have;

$$|H \cap C^g| = \begin{cases} 3 \; if \; g \in \tau_1 \\ 3 \; if \; g \in \tau_2. \\ 3 \; if \; g \in \tau_0 \end{cases}$$

If $p = 3$, G contains $(q-1)(q+1)$ conjugate elements of order 3. If $p \neq 3$, there is a single conjugacy class of elements of order 3 in G containing $q(q+\partial)$ elements where,

$$\partial = \begin{cases} 1 \; if \; q \equiv 1 \; mod 3 \\ -1 \; if \; q \equiv -1 \; mod 3 \end{cases}.$$

Therefore we have,

$$|H \cap C^g| = \begin{cases} 8 \; if \; g \in \tau_1 \\ 8 \; if \; g \in \tau_2. \\ 8 \; if \; g \in \tau_0 \end{cases}$$

(Kamuti, 1992, p 60)

Now we have the following four cases to consider;

a) $p = 2, \quad q \equiv 1 \; mod \; 3$

b) $p = 3$

c) $p > 3, \quad q \equiv 1 \; mod \; 3$

d) $p > 3, \quad q \equiv -1 \; mod \; 3$

Since we have obtained $|C^g|$ and $|H \cap C^g|$, we can now proceed to compute $\pi(g)$ in all the four cases. These are displayed in Table 3.4.2 below.

Table 3.4.2: No. of fixed points of elements of *G* acting on the cosets of $\boldsymbol{A_4}$

		$\|C^g\|$	$\|C^g \cap H\|$	$\pi(g)$
$g \in \tau_1$	case a) Case b), $d = 3$ Case c) and d)	$q^2 - 1$ $q^2 - 1$ $q^2 - 1$	3 8 0	2^{f-2} $\frac{2}{3}q$ 0
$g \in \tau_2$	Case a) and c) $d = 3$	$q(q+1)$	8	$\frac{2}{3}(q-1)$
	Case b), c) and d) , $d = 2$	$\frac{q(q+1)}{2}$	$\begin{cases} 3 \; if \; p^f, f \; even \\ 0 \; if \; p^f, f \; odd \end{cases}$	$\begin{cases} \frac{q-1}{2} \\ 0 \end{cases}$
	Case a) - d) d$\neq$ 2,3	$q(q+1)$	0	0
$g \in \tau_0$	Case b), c) and d), $d = 2$	$\frac{q(q-1)}{2}$	$\begin{cases} 3 \; if \; p^f, f \; odd \\ 0 \; if \; p^f, f \; even \end{cases}$	$\begin{cases} \frac{q+1}{2} \\ 0 \end{cases}$
	Case d) $d = 3$	$q(q-1)$	8	$\frac{2}{3}(q+1)$
	Case a) – d) d$\neq$ 2,3	$q(q-1)$	0	0

Here we give case c) with $g \in \tau_2$ as an example of how we obtain the number α_l and use similar approach for the other 3 cases.

This case splits into the following four subcases;

i) $2|d$ and $3 \nmid d$ ii) $3|d$ and $2 \nmid d$ iii) $2, 3|d$ iv) $2,3 \nmid d$

We only work out subcase iii);

Using arguments similar to those in Section 3.1 it can be shown that;

When $l = \frac{d}{2}$,

$$\alpha_{\frac{d}{2}} = \frac{1}{d}(q-1).$$

When $l = \frac{d}{3}$,

$$\alpha_{\frac{d}{3}} = \frac{2}{d}(q-1).$$

When $l = d$,

$$\alpha_d = \frac{1}{d}\left[\frac{q(q^2-1)}{12} - \frac{2}{3}(q-1) - \frac{(q-1)}{2}\right]$$

$$= \frac{1}{12d}[(q^2+q-14)(q-1)].$$

The disjoint cycle structures of elements of G on the cosets of A_4 are as shown in below.

Summary of the results

$g \in \tau_1$

Case a) g' contains $2^{f-2} = \frac{q}{4}$ 1-cycle and $\frac{q}{24}(q^2-4)$ p-cycles

Case b) g' contains $\frac{2}{3}q$ 1-cycle and $\frac{q}{36}(q^2-9)$ p-cycles

Case c) and Case d) g' contains $\frac{q}{p}(q^2-1)$ p-cycles

Results for $g \in \tau_2$ and $g \in \tau_0$ are displayed in table 3.4.3 below.

Table 3.4.3: Disjoint cycle structures of elements of *G* on the cosets of $\boldsymbol{A_4}$

$g \in \tau_2$

Cycle length of g'		$\frac{d}{2}$	$\frac{d}{3}$	d
No. of Cycles				
Case a)	$3\|d$	0	$\frac{2}{d}(q-1)$	$\left[\frac{(q^2+q-8)(q-1)}{12d}\right]$
	$3\nmid d$	0	0	$\left[\frac{q(q^2-1)}{12d}\right]$
Case b) and d)	$2\|d$	$\frac{1}{d}(q-1)$	0	$\left[\frac{(q-2)(q+3)(q-1)}{12d}\right]$
	$2\nmid d$	0	0	$\left[\frac{q(q^2-1)}{12d}\right]$
Case c)	$3\|d$	0	$\frac{2}{d}(q-1)$	$\left[\frac{(q^2+q-8)(q-1)}{12d}\right]$
	$2\|d$	$\frac{1}{d}(q-1)$	0	$\left[\frac{(q-2)(q+3)(q-1)}{12d}\right]$
	$2,3\|d$	$\frac{1}{d}(q-1)$	$\frac{2}{d}(q-1)$	$\left[\frac{(q^2+q-14)(q-1)}{12d}\right]$
	$2,3\nmid d$	0	0	$\left[\frac{q(q^2-1)}{12d}\right]$

$g \in \tau_0$

Cycle length of g'		$\frac{d}{2}$	$\frac{d}{3}$	d
No. of Cycles				
Case a)	$2,3 \nmid d$	0	0	$\left[\frac{q(q^2-1)}{12d}\right]$
Case b) and c)	$2\|d$	$\frac{1}{d}(q+1)$	0	$\left[\frac{(q+2)(q-3)(q+1)}{12d}\right]$
	$2 \nmid d$	0	0	$\left[\frac{q(q^2-1)}{12d}\right]$
Case d)	$3\|d$	0	$\frac{2}{d}(q+1)$	$\left[\frac{(q^2-q-8)(q+1)}{12d}\right]$
	$2\|d$	$\frac{1}{d}(q+1)$	0	$\left[\frac{(q+2)(q-3)(q+1)}{12d}\right]$
	$2,3\|d$	$\frac{1}{d}(q+1)$	$\frac{2}{d}(q+1)$	$\left[\frac{(q^2-q-14)(q+1)}{12d}\right]$
	$2,3 \nmid d$	0	0	$\left[\frac{q(q^2-1)}{12d}\right]$

3.5 Representations on the cosets of $H = A_5$

G has a subgroup H isomorphic to A_5 if $p = 5$ or $q \equiv \pm 1 \; mod\; 5$. Together with the identity element, H contains 24 elements of order 5 forming two conjugacy classes of 12 elements which are transposed by squaring, 20 conjugate elements of order 3 and 15 conjugate elements of order 2. We can summarize this information in Table 3.5.1 below.

Table 3.5.1 Cycle structures of elements of $\boldsymbol{A_5}$

Cycle structures	No. of permutation	order
I	1	1
(ab)(cd)	15	2
(abc)	20	3
(abcde)	24	5

There are 15 involutions in *H*. So every conjugacy class of elements of order 2 if $p \neq 2$ in *G* satisfies;

$$|H \cap C^g| = 15\;, if\; p = 2.$$

$$|H \cap C^g| = 15\; or\; 0\;, if\; p \neq 2.$$

There is a single conjugacy class containing 20 elements of order 3 in *H*. If $p \neq 3$, we have;

$$|H \cap C^g| = 20.$$

If $p = 3$, then *f* is even (since $q \equiv \pm 1\; mod\; 5$), so the class of elements of order 3 in *G* satisfies;

$$|H \cap C^g| = 20\; or\; 0.$$

(Kamuti, 1992 p. 60)

If $p = 5$, there are $q^2 - 1$ elements of order 5 in *G*. These elements form two self-inverse conjugacy classes containing $\frac{q^2-1}{2}$ elements. Squaring preserves or transposes these elements as *f* is even or odd. If $q \equiv \pm 1 \ mod\ 5$ there are two self-inverse conjugacy classes of $q(q + \partial)$ elements, where

$$\partial = \begin{cases} 1\ if\ q \equiv 1\ mod\ 5 \\ -1\ if\ q \equiv -1\ mod\ 5 \end{cases}.$$

Squaring transposes the two classes.

Hence if $d = 5$, then

$$|H \cap C^g| = \begin{cases} 24\ or\ 0 \quad if\ \ p = 5, f\ even \\ 12\ if\ p = 5, f\ odd\ or\ q \equiv \pm 1\ mod\ 5. \end{cases}$$

Now the following are the cases to consider:

a) $p = 2$ $\quad$ $q \equiv 4\ mod\ 15$

b) $p = 2$ $\quad$ $q \equiv 11\ mod\ 15$

c) $p = 2$ $\quad$ $q \equiv 1 mod\ 15$

d) $p = 2$ $\quad$ $q \equiv -1\ mod\ 15$

e) $p = 3$ $\quad$ $q \equiv 1\ mod\ 5$

f) $p = 3$ $\quad$ $q \equiv -1\ mod\ 5$

g) $p = 5$ $\quad$ $q \equiv 1\ mod\ 3$

h) $p = 5$ $\quad$ $q \equiv -1\ mod\ 3$

i) $p > 5$ $\quad$ $q \equiv -1\ mod\ 15$

j) $p > 5$ $\quad$ $q \equiv 11\ mod\ 15$

k) $p > 5$ $\quad$ $q \equiv 1\ mod\ 15$

l) $p > 5$ $\quad$ $q \equiv 19\ mod\ 30$

Since we have obtained $|\boldsymbol{C^g}|$ and $|\boldsymbol{H} \cap \boldsymbol{C^g}|$ we now proceed to compute $\boldsymbol{\pi(g)}$ in all the cases shown above. The results are displayed in Table 3.5.2 below.

Table 3.5.2: No of fixed points of elements *G* acting on the cosets of $\boldsymbol{A_5}$

		$\|C^g\|$	$\|H \cap C^g\|$	$\pi(g)$
I $g \in \tau_1$	Case a) , b), c), d) $p = 2$	$q^2 - 1$	15 or 0	2^{f-2}
	Case e), f)	$q^2 - 1$	20 or 0	3^{f-1}
	Case g), h)	$q^2 - 1$	12	(5^{f-1})
	Case i), j), k), l)	$q^2 - 1$	0	0
II $g \in \tau_2$	Case a), c), g), k), l) $d = 3$	$q(q+1)$	$\begin{cases} 20 \text{ } if \ p^f, f \ \ even \\ 0 \text{ } if \ \ p^f, f \ \ odd \end{cases}$	$\begin{cases} \frac{q-1}{3} \\ 0 \end{cases}$
	Case b), c), e), j), k) $d = 5$	$q(q+1)$	12	$\frac{(q-1)}{5}$
	Case e), f), g), h), i), j), k), l) $d = 2$	$\frac{q(q+1)}{2}$	$\begin{cases} 15 \text{ } if \ p^f, f \ \ even \\ 0 \text{ } if \ \ p^f, f \ \ odd \end{cases}$	$\begin{cases} \frac{q-1}{2} \\ 0 \end{cases}$
	Case $a)$ -l) $d \neq 2,3,5$	$q(q+1)$	0	0
III $g \in \tau_0$	Case a), d), f), i), l) $d = 5$	$q(q-1)$	12	$\frac{(q+1)}{5}$
	Case b), d), h), i), j) $d = 3$	$q(q-1)$	$\begin{cases} 20 \text{ } if \ p^f, f \ \ odd \\ 0 \text{ } if \ \ p^f, f \ \ even \end{cases}$	$\begin{cases} \frac{q+1}{3} \\ 0 \end{cases}$
	Case e), f), g), h), i), j), k), l) $d = 2$	$\frac{q(q-1)}{2}$	$\begin{cases} 15 \text{ } if \ p^f, f \ \ odd \\ 0 \text{ } if \ \ p^f, f \ \ even \end{cases}$	$\begin{cases} \frac{q+1}{2} \\ 0 \end{cases}$
	Case $a)$ -l) $d \neq 2,3,5$	$q(q-1)$	0	0

Here we give case k) with $g \in \tau_2$ as an example of how we obtain the number α_l and use similar approach for the other cases.

We have the following subcases:

i) $2|d$ and $5, 3 \nmid d$ ii) $3|d$ $5, 2 \nmid d$ iii) $5|d$ and $2, 3 \nmid d$

iv) $2,3|d$, and $5 \nmid d$ v) $3,5|d$, $2 \nmid d$ vi) $2, 5|d$ and $3 \nmid d$

vii) $2, 3, 5|d$ viii) $2, 3, 5 \nmid d$

We only work out subcase (vii)

By using similar arguments to those in section 3.1 it can be shown that

$$\alpha_{\frac{d}{2}} = \frac{1}{d}(q-1), \ \alpha_{\frac{d}{3}} = \frac{1}{d}(q-1), \alpha_{\frac{d}{5}} = \frac{1}{d}(q-1), \quad \alpha_d = \left[\frac{(q-1)(q^2+q-62)}{60d}\right].$$

The disjoint cycle structures of elements of G on the cosets of A_5 are shown below.

Summary of the results

$g \in \tau_1$

Case a) – d), g' contains $2^{f-2} = \frac{q}{4}$ 1-cycle and $\frac{q}{60p}(q^2-16)$ p-cycles

Case e) and f) g' contains $3^{f-1} = \frac{1}{3}q$ 1-cycle and $\frac{q}{60p}(q^2-21)$ p-cycles

Case g) and h) g' contains $(5^{f-1}) = \frac{1}{5}q$ 1-cycle and $\frac{q}{60p}(q^2-25)$ p-cycles

Case i) - l) g' contains $\frac{q}{60p}(q^2-1)$ p-cycles

Results for $g \in \tau_2$ and $g \in \tau_0$ are displayed in Table 3.5.3 below.

Table 3.5.3: Disjoint Cycle structures of elements of G on the cosets of $\boldsymbol{A_5}$

$g \in \tau_2$

Cycle length of g'		$\frac{d}{2}$	$\frac{d}{3}$	$\frac{d}{5}$	d
No. of Cycles					
Case a) and case l)	$3\|d$	0	$\frac{1}{d}(q-1)$	0	$\left[\frac{(q-1)(q+5)(q-4)}{60d}\right]$
	$3 \nmid d$	0	0		$\left[\frac{q(q^2-1)}{60d}\right]$
Case b)	$5\|d$	$\frac{1}{d}(q-1)$	0	0	$\left[\frac{(q-1)(q+4)(q-3)}{60d}\right]$
	$5 \nmid d$	0	0	0	$\left[\frac{q(q^2-1)}{60d}\right]$
Case c)	$3\|d$	0	$\frac{1}{d}(q-1)$	0	$\frac{(q-1)(q+5)(q-4)}{60d}$
	$5\|d$	0	0	$\frac{1}{d}(q-1)$	$\left[\frac{(q-1)(q+4)(q-3)}{60d}\right]$
	$5,3\|d$	0	$\frac{1}{d}(q-1)$	$\frac{1}{d}(q-1)$	$\left[\frac{(q-1)(q^2+q-32)}{60d}\right]$
	$5,3 \nmid d$	0	0	0	$\left[\frac{q(q^2-1)}{60d}\right]$
Case d	$5,3 \nmid d$	0	0	0	$\left[\frac{q(q^2-1)}{60d}\right]$
Case	$2\|d$	$\frac{1}{d}(q-1)$	0	0	$\frac{(q-1)(q+6)(q-5)}{60d}$

e) & j)	$5\|d$	0	0	$\frac{1}{d}(q-1)$	$\left[\frac{(q-1)(q+4)(q-3)}{60d}\right]$
	$5,2\|d$	$\frac{1}{d}(q-1)$	0	$\frac{1}{d}(q-1)$	$\left[\frac{(q-1)(q^2+q-42)}{60d}\right]$
	$5,2\nmid d$	0	0	0	$\left[\frac{q(q^2-1)}{60d}\right]$
Case f) and i)	$2\|d$	$\frac{1}{d}(q-1)$	0	0	$\frac{(q-1)(q+6)(q-5)}{60d}$
	$2\nmid d$	0	0	0	$\left[\frac{q(q^2-1)}{60d}\right]$
Case g)	$2\|d$	$\frac{1}{d}(q-1)$	0	0	$\frac{(q-1)(q+6)(q-5)}{60d}$
	$3\|d$	0	$\frac{1}{d}(q-1)$	0	$\left[\frac{(q-1)(q+5)(q-4)}{60d}\right]$
	$3,2\|d$	$\frac{1}{d}(q-1)$	$\frac{1}{d}(q-1)$	0	$\left[\frac{(q-1)(q^2+q-50)}{60d}\right]$
	$3,2\nmid d$	0	0	0	$\left[\frac{q(q^2-1)}{60d}\right]$
Case k)	$2\|d$	$\frac{1}{d}(q-1)$	0	0	$\frac{(q-1)(q+6)(q-5)}{60d}$
	$3\|d$	0	$\frac{1}{d}(q-1)$	0	$\left[\frac{(q-1)(q+5)(q-4)}{60d}\right]$
	$5\|d$	0	0	$\frac{1}{d}(q-1)$	$\left[\frac{(q-1)(q+4)(q-3)}{60d}\right]$
	$3,2\|d$	$\frac{1}{d}(q-1)$	$\frac{1}{d}(q-1)$	0	$\left[\frac{(q-1)(q^2+q-50)}{60d}\right]$

	$5,2\|d$	$\frac{1}{d}(q-1)$	0	$\frac{1}{d}(q-1)$	$\left[\frac{(q-1)(q^2+q-42)}{60d}\right]$
	$5,3\|d$	0	$\frac{1}{d}(q-1)$	$\frac{1}{d}(q-1)$	$\left[\frac{(q-1)(q^2+q-32)}{60d}\right]$
	$5,2,3\|d$	$\frac{1}{d}(q-1)$	$\frac{1}{d}(q-1)$	$\frac{1}{d}(q-1)$	$\left[\frac{(q-1)(q^2+q-62)}{60d}\right]$
	$5,3,2 \nmid d$	0	0	0	$\left[\frac{q(q^2-1)}{60d}\right]$

$$g \in \tau_0$$

Cycle length of g'		$\frac{d}{2}$	$\frac{d}{3}$	$\frac{d}{5}$	d
No. of Cycles Case a)					
	$5\|d$	0	0	$\frac{1}{d}(q+1)$	$\left[\frac{(q+1)(q-4)(q+3)}{60d}\right]$
	$5 \nmid d$	0	0	0	$\left[\frac{q(q^2-1)}{60d}\right]$
Case b) and j)	$3\|d$	0	$\frac{1}{d}(q+1)$	0	$\left[\frac{(q+4)(q-5)(q+1)}{60d}\right]$
	$3 \nmid d$	0	0	0	$\left[\frac{q(q^2-1)}{60d}\right]$

Case c	$2,3,5 \nmid d$	0	0	0	$\left[\frac{q(q^2-1)}{60d}\right]$
Case d)	$3\|d$	0	$\frac{2}{d}(q+1)$	0	$\left[\frac{(q+4)(q-5)(q+1)}{60d}\right]$
	$5\|d$	0	0	$\frac{1}{d}(q+1)$	$\left[\frac{(q+1)(q-4)(q+3)}{60d}\right]$
	$5,3\|d$	0	$\frac{1}{d}(q+1)$	$\frac{1}{d}(q+1)$	$\left[\frac{(q^2-q-32)(q+1)}{12d}\right]$
	$5,3 \nmid d$	0	0	0	$\left[\frac{q(q^2-1)}{60d}\right]$
Case e), g), and k)	$2\|d$	$\frac{1}{d}(q+1)$	0	0	$\frac{(q+1)(q-6)(q+5)}{60d}$
	$2 \nmid d$	0	0	0	$\left[\frac{q(q^2-1)}{60d}\right]$
Case f) and l)	$2\|d$	$\frac{1}{d}(q+1)$	0	0	$\frac{(q+1)(q-6)(q+5)}{60d}$
	$5\|d$	0	0	$\frac{1}{d}(q+1)$	$\left[\frac{(q+1)(q-4)(q+3)}{60d}\right]$
	$5,2\|d$	$\frac{1}{d}(q+1)$	0	$\frac{1}{d}(q+1)$	$\left[\frac{(q+1)(q^2-q-42)}{60d}\right]$
	$5,2 \nmid d$	0	0	0	$\left[\frac{q(q^2-1)}{60d}\right]$
Case h)	$3\|d$	0	$\frac{1}{d}(q+1)$	0	$\left[\frac{(q+4)(q-5)(q+1)}{60d}\right]$
	$2\|d$	$\frac{1}{d}(q+1)$	0	0	$\frac{(q+1)(q-6)(q+5)}{60d}$
	$3,2\|d$	$\frac{1}{d}(q+1)$	$\frac{1}{d}(q+1)$	0	$\left[\frac{(q+1)(q^2-q-50)}{60d}\right]$

	$3,2 \nmid d$	0	0	0	$\left[\frac{q(q^2-1)}{60d}\right]$
Case i)	$2\|d$	$\frac{1}{d}(q+1)$	0	0	$\frac{(q+1)(q-6)(q+5)}{60d}$
	$3\|d$	0	$\frac{1}{d}(q+1)$	0	$\left[\frac{(q+1)(q-5)(q+4)}{60d}\right]$
	$5\|d$	0	0	$\frac{1}{d}(q+1)$	$\left[\frac{(q+1)(q-4)(q+3)}{60d}\right]$
	$3,2\|d$	$\frac{1}{d}(q+1)$	$\frac{1}{d}(q+1)$	0	$\left[\frac{(q+1)(q^2-q-50)}{60d}\right]$
	$5,2\|d$	$\frac{1}{d}(q+1)$	0	$\frac{1}{d}(q+1)$	$\left[\frac{(q+1)(q^2-q-42)}{60d}\right]$
	$5,3\|d$	0	$\frac{1}{d}(q+1)$	$\frac{1}{d}(q+1)$	$\left[\frac{(q+1)(q^2-q-32)}{60d}\right]$
	$5,2,3\|d$	$\frac{1}{d}(q+1)$	$\frac{1}{d}(q+1)$	$\frac{1}{d}(q+1)$	$\left[\frac{(q+1)(q^2-q-62)}{60d}\right]$
	$5,3,2 \nmid d$	0	0	0	$\left[\frac{q(q^2-1)}{60d}\right]$

3.6 Representations on the cosets of $H = D_{2(q-1)}$

If C_{q-1} is the maximal cyclic subgroup of H, then the $q-1$ involutions in $H\backslash C_{q-1}$ lie in two conjugacy classes of $\frac{q-1}{2}$ in H; one lying entirely in $PSL(2,q)$, the other entirely in $G\backslash PSL(2,q)$. In total H has $\frac{q-1}{2}$ elements of order 2 if $g \in \tau_2$.

Let $< s > = C_{q-1}$. Then the conjugacy class of $s^j \; j \in \mathbb{N}$ in H is $\{s^j, s^{-j}\}$. So every element and its inverse are in one conjugacy class. Involution in G form a single conjugacy class containing

$$\begin{cases} \frac{q(q+1)}{2} & if \; g \in \tau_2 \\ \frac{q(q-1)}{2} & if \; g \in \tau_0 \\ q^2 - 1 & if \; g \in \tau_1 \end{cases}$$

elements.

If $d > 2$, then $|C^g|$ in G is;

$$\begin{cases} q(q+1) & if \; g \in \tau_2 \\ q(q-1) & if \; g \in \tau_0. \\ q^2 - 1 & if \; g \in \tau_1 \end{cases}$$

From the above information we have;

$$|C^g \cap H| = \begin{cases} \frac{q+1}{2} & if \; g \in \tau_2 \\ \frac{q-1}{2} & if \; g \in \tau_0 \\ q - 1 & if \; g \in \tau_1 \end{cases}$$

if $d = 2$.

If $d > 2$, then

$$|C^g \cap H| = \begin{cases} 0 \; if \; g \in \tau_0 \; 0r \; g \in \tau_1 \\ 2 \;\; if \;\; g \in \tau_2 \end{cases}.$$

So we must consider two cases

a) When q is odd

b) When q is even

Case a) When q is odd

Since we have obtain $|\boldsymbol{C^g}|$ and $|\boldsymbol{H} \cap \boldsymbol{C^g}|$ we now proceed to compute $\boldsymbol{\pi}(\boldsymbol{g})$. The results are displayed in Table 3.6.1 below.

Table 3.6.1: No. of fixed points of G acting on the cosets of $\boldsymbol{D_{2(q-1)}}$ when q is odd

Case			$\|C^g\|$	$\|C^g \cap H\|$	$\pi(g)$
I	$g \in \tau_0$	$d > 2$	$q(q-1)$	0	0
		$d = 2$	$\frac{q(q-1)}{2}$	$\frac{q-1}{2}$	$\frac{q+1}{2}$
II	$g \in \tau_1$	$d > 2$	q^2-1	0	0
III	$g \in \tau_2$	$d > 2$	$q(q+1)$	2	1
		$d = 2$	$\frac{q(q+1)}{2}$	$\frac{q+1}{2}$	$\frac{q+1}{2}$

By using arguments similar to those in section 3.1, the disjoint cycle structures of elements of G on the cosets of $D_{2(q-1)}$ are shown below.

Summary of results

$g \in \tau_1$

g' contains $p^{f-1}\left(\frac{q+1}{2}\right)$ p-cycles

Table 3.6.2: Disjoint Cycle structures of elements of *G* on the cosets of $\boldsymbol{D_{2(q-1)}}$ when *q* is odd

		τ_2			τ_0	
Cycle length of g'		1	$\frac{d}{2}$	d	$\frac{d}{2}$	d
No. of cycles	d even	1	$\frac{q-1}{d}$	$\frac{1}{2d}(q^2-1)$	$\frac{q+1}{d}$	$\frac{1}{2d}(q^2-1)$
	d odd	1	0	$\frac{(q-1)(q+2)}{2d}$	0	$\frac{q(q+1)}{2d}$

Case b) When *q* is even

From the information given above we obtain the following table.

Table 3.6.3: No of fixed points of *G* acting on the cosets of $\boldsymbol{D_{2(q-1)}}$ when *q* is even

Case		$\lvert C^g \rvert$	$\lvert C^g \cap H \rvert$	$\pi(g)$
I	$g \in \tau_0$	$\frac{q(q-1)}{2}$	0	0
II	$g \in \tau_1$	q^2-1	$q-1$	$\frac{q}{2}$
III	$g \in \tau_2$	$\frac{q(q+1)}{2}$	2	1

NB: Applying the same argument as in Section 3.1 above we obtain the disjoint cycle structures of elements of *G* on the cosets of of $\boldsymbol{D_{2(q-1)}}$ as shown in Table 3.6.2 below..

Summary of results

$g \in \tau_1$

g' contains 2^{f-1} 1-cycle and 4^{f-1} p-cycles

Table 3.6.4: Disjoint Cycle structures of elements of G on the cosets of $\boldsymbol{D_{2(q-1)}}$ when q is even

		τ_2			τ_0	
Cycle length of g'		1	$\frac{d}{2}$	d	$\frac{d}{2}$	d
No. of cycles	d even	1	0	$\frac{1}{2d}(q^2-1)$	0	$\frac{1}{2d}q(q+1)$

CHAPTER FOUR

THE CYCLE INDEX FORMULAS FOR $G = PGL(2, q)$

ACTING ON THE COSETS OF ITS SUBGROUPS

After computing the disjoint cycle structures of elements G acting on the cosets of its subgroups, then we can now use them to find the cycle index formula for these representations.

In this chapter we shall determine some general formulas for finding the cycle indices for the representation of G on the cosets of its subgroups. This chapter is divided into six sections. In Section 4.1 to Section 4.6 we find the cycle index formula for the representations of G on the cosets of $C_{q-1}, C_{q+1}, P_q, A_4, A_5$ and $D_{2(q-1)}$ respectively. In each section we give an example.

4.1 Cycle index of G acting on the cosets of $H = C_{q-1}$

Before finding the cycle index formula of *G* acting on the cosets of *H* we first give a theorem by Redfield which will be used in this section.

4.1.1 Theorem

The cycle index of the regular representation of a cyclic group C_n is given by;

$$Z(C_n) = \frac{1}{n}\sum_{d|n} \varphi(d) t_d^{n/d},$$

where φ is the Euler's Phi function and $t_{1,}t_{2,} \dots \dots \dots, t_n$ are distinct (commuting) indeterminates.

Proof: For proof see Redfield, 1927. ■

4.1.2 Theorem

The cycle index of G on the cosets of H is given by;

$$Z(G) = \frac{1}{|G|}\left[t_1^{|G|/|H|} + (q^2-1)t_p^{p^{f-1}(q+1)}\right.$$
$$+ \frac{q(q+1)}{2}\sum_{1\neq d|q-1}\varphi(d)t_1^2 t_d^{\frac{(q-1)(q+2)}{d}}$$
$$\left. + \frac{q(q-1)}{2}\sum_{1\neq d|q+1}\varphi(d)t_d^{\frac{q(q+1)}{d}}\right].$$

Proof

The identity contributes $t_1^{|G|/|H|}$ to the sum of the monomials. All the (q^2-1) parabolics lie in the same conjugacy class, hence they have the same monomials. So from Table 3.1.2 the contributions by elements of τ_1 is $(q^2-1)t_p^{p^{f-1}(q+1)}$. Each $g \in \tau_2$ is contained in a unique cyclic subgroup C_{q-1} and there are in total $\frac{q(q+1)}{2}$ conjugates of C_{q-1}. Hence by Theorem 4.1.1 and the results in Table 3.1.2 the contributions by elements of τ_2 is $\frac{q(q+1)}{2}\sum_{1\neq d|q-1}\varphi(d)t_1^2 t_d^{\frac{(q-1)(q+2)}{d}}$. Finally each $g \in \tau_0$ is contained in unique cyclic subgroup C_{q+1} and there are in total $\frac{q(q-1)}{2}$ conjugates of C_{q+1} so they contribute

$\frac{q(q-1)}{2}\sum_{1\neq d|q+1}\varphi(d)t_d^{\frac{q(q+1)}{d}}$ to the same of the monomial. Adding all the contributions and dividing by the order of G we get the desired results. ■

4.1.3 Examples

4.1.3.1 Example

From Theorem 4.1.2 above the cycle index of $G = PGL(2,2)$ acting on the cosets of H is;

$$Z(G) = \frac{1}{6}[t_1^6 + 3t_2^3 + 2t_3^2].$$

This cycle index is the same as the cycle index of S_3 acting on itself by left multiplication as expected since $PGL(2,2) \cong S_3$.

4.1.3.2 Example

The cycle index of $G = PGL(2,3)$ acting on the cosets of H is given by;

$$Z(G) = \frac{1}{24}[t_1^{12} + 8t_3^4 + 6t_1^2t_2^5 + 3t_2^6 + 6t_4^3].$$

This is the same as the cycle index of S_4 acting on the cosets of its subgroup of order 2 as expected since $PGL(2,3) \cong S_4$.

4.1.3.3 Example

The cycle index of $G = PGL(2,4)$ acting on the cosets of H is given by;

$$Z(G) = \frac{1}{60}[t_1^{20} + 15t_2^{10} + 20t_1^2t_3^6 + 24t_5^4].$$

This is the same as the cycle index of A_5 acting on the cosets of its subgroups of order 3 as expected since $PGL(2,4) \cong A_5$.

4.2 Cycle index of G acting on the cosets of $H = C_{q+1}$

4.2.1 Theorem

The cycle index of $PGL(2,q)$ acting on the cosets of H is given by;

$$Z(G) = \frac{1}{|G|}\left[t_1^{|G|/|H|} + (q^2-1)t_p^{p^{f-1}(q-1)} + \frac{q(q+1)}{2}\sum_{1\neq d|q-1}\varphi(d)t_d^{\frac{q(q-1)}{d}} + \frac{q(q-1)}{2}\sum_{1\neq d|q+1}\varphi(d)t_1^2 t_d^{\frac{(q+1)(q-2)}{d}}\right].$$

Proof

Using the results in Table 3.2.2 and argument similar to those in Theorem 4.1.2 the result is immediate. ■

4.2.2 Examples

4.2.2.1 Example

From Theorem 4.2.1 above the cycle index of $G = PGL(2,2)$ on the cosets of H is given by;

$$Z(G) = \frac{1}{6}[t_1^2 + 3t_2 + 2t_1^2].$$

This cycle index is the same as the cycle index of S_3 acting on the cosets of its subgroup of order 3 as expected since $PGL(2,2) \cong S_3$.

4.2.2.2Example

Again from Theorem 4.2.1 the cycle index of $G = PGL(2,3)$ acting on the cosets of H is given by;

$$Z(G) = \frac{1}{24}[t_1^6 + 8t_3^2 + 6t_2^3 + 3t_1^2t_2^2 + 6t_2t_4].$$

This is the same as the cycle index of S_4 acting on the cosets of its subgroup of order 4 as expected since $PGL(2,3) \cong S_4$.

4.2.2.3 Example

The cycle index of $G = PGL(2,4)$ acting on the cosets of $H = C_{q+1}$ is given by;

$$Z(G) = \frac{1}{60}[t_1^{12} + 15t_2^6 + 20t_3^4 + 24t_1^2t_5^2].$$

This is the same as the cycle index of A_5 acting on the cosets of its subgroup of order 5 as expected since $PGL(2,4) \cong A_5$.

4.3 Cycle index of G acting on the cosets of $H = P_q$

From the results in Table 3.3.2, we have the following theorem;

4.3.1 Theorem

The cycle index of G on the cosets of H is given by;

$$Z(G) = \frac{1}{|G|}\left[t_1^{|G|/|H|} + (q^2-1)t_1^{q-1}t_p^{p^{f-1}(q-1)} + \frac{q(q+1)}{2}\sum_{1\neq d|q-1}\varphi(d)t_d^{\frac{1}{d}(q^2-1)} + \frac{q(q-1)}{2}\sum_{1\neq d|q+1}\varphi(d)t_d^{\frac{1}{d}(q^2-1)}\right].$$

4.3.2 Examples

4.3.2.1 Example

Cycle index of $PGL(2,2)$ acting on the cosets of H is given by;

$$Z(G) = \frac{1}{6}[t_1^3 + 3t_1t_2 + 2t_3].$$

This is the same as the cycle index of S_3 acting on the cosets of its subgroup of order 2 as expected since $PGL(2,2) \cong S_3$.

4.3.2.2 Example

Again from Theorem 4.3.1 the cycle index of $PGL(2,3)$ acting on the cosets of H is given by;

$$Z(G) = \frac{1}{24}[t_1^8 + 8t_1^2t_3^2 + 6t_2^4 + 3t_2^4 + 6t_4^2].$$

This is the same as the cycle index of S_4 acting on the cosets of its subgroup of order 3 as expected since $PGL(2,3) \cong S_4$.

4.4.2.3 Example

The cycle index of $G = PGL(2,4)$ acting on the cosets of H is given by;

$$Z(G) = \frac{1}{60}[t_1^{15} + 15t_1^3t_2^6 + 20t_3^5 + 24t_5^3].$$

This is the same as the cycle index of A_5 acting on the cosets of its subgroup of order 4 as expected since $PGL(2,4) \cong A_5$.

4.4 Cycle index of G acting on the cosets of $H = A_4$

The cycle index of G acting on the cosets of H depends on the cases given in Section 3.4. We are going to prove case (a) and just state the other cases.

4.4.1 Theorem

Case a)

Cycle index of G acting on the cosets of H is given by;

$$Z(G) = \frac{1}{|G|}\left[t_1^{\frac{q(q^2-1)}{12}} + (q^2-1)t_1^{\frac{q}{4}} t_p^{\frac{q(q^2-4)}{24}} \right.$$

$$+ \frac{q(q+1)}{2}[\sum_{3|d|q-1} \varphi(d) t_{\frac{d}{3}}^{\frac{2(q-1)}{d}} t_d^{\frac{1}{12d}(q^2+q-8)(q-1)}$$

$$\left. + \sum_{3\nmid d|q-1} \varphi(d) t_d^{\frac{1}{12d}q(q^2-1)}] + \frac{q(q-1)}{2}[\sum_{d|q+1} \varphi(d) t_d^{\frac{1}{12d}q(q^2-1)}] \right].$$

Proof

From the results in Table 3.4.3 the contribution to the sum of the monomials by the identity is $t_1^{\frac{q(q^2-1)}{12}}$. The contribution by elements of τ_1 is $(q^2-1)t_1^{\frac{q}{4}} t_p^{\frac{q(q^2-4)}{24}}$.

We have two different types of monomials for elements of τ_2;

i) $t_{\frac{d}{3}}^{\frac{2(q-1)}{d}} t_d^{\frac{1}{12d}(q^2+q-8)(q-1)}$, if $3|d|q-1$.

ii) $t_d^{\frac{1}{12d}q(q^2-1)}$, if $2,3 \nmid d|q-1$.

Hence elements of τ_2 contribute;

$$\frac{q(q+1)}{2}\left[\sum_{3|d|q-1}\varphi(d)t_{\frac{d}{3}}^{\frac{2(q-1)}{d}}t_d^{\frac{1}{12d}(q^2+q-8)(q-1)}+\sum_{2,3\nmid d|q-1}\varphi(d)t_d^{\frac{1}{12d}q(q^2-1)}\right].$$

Elements of τ_0 contributes $\frac{q(q-1)}{2}[\sum_{d|q+1}\varphi(d)t_d^{\frac{1}{12d}q(q^2-1)}$.

Adding all the above contributions and dividing by the order of G gives the above results. ■

4.4.2 Example

The cycle index of $G = PGL(2,4)$ acting on the cosets of H is given by;

$$Z(G) = \frac{1}{60}[t_1^5 + 15t_1t_2^2 + 20t_3t_1^2 + 24t_5^1].$$

This is the same as the cycle index of A_5 acting on the cosets of its subgroups of order 12 as expected since $PGL(2,4) \cong A_5$.

Case b)

The cycle index of G acting on the cosets of H is given by;

$$Z(G) = \frac{1}{|G|}\left[t_1^{\frac{q(q^2-1)}{12}} + (q^2-1)t_1^{\frac{2q}{3}}t_p^{\frac{q(q^2-9)}{12p}}\right.$$
$$+\frac{q(q+1)}{2}[\sum_{2|d|q-1}\varphi(d)t_{\frac{d}{2}}^{\frac{q-1}{d}}t_d^{\frac{1}{12d}(q-2)(q+3)(q-1)}$$
$$+\sum_{2\nmid d|q-1}\varphi(d)t_d^{\frac{1}{12d}q(q^2-1)}]$$
$$+\frac{q(q-1)}{2}[\sum_{2|d|q+1}\varphi(d)t_{\frac{d}{2}}^{\frac{q+1}{d}}t_d^{\frac{1}{12d}(q+2)(q-3)(q+1)}$$
$$\left.+\sum_{d|q+1}\varphi(d)t_d^{\frac{1}{12d}q(q^2-1)}]\right].$$

Case c)

The cycle index of G acting on the cosets of H is given by;

$$Z(G) = \frac{1}{|G|}\left[t_1^{\frac{q(q^2-1)}{12}} + (q^2-1)t_p^{\frac{q(q^2-1)}{12p}}\right.$$

$$+\frac{q(q+1)}{2}[\sum_{2|d|q-1}\varphi(d)t_{\frac{d}{2}}^{\frac{q-1}{d}}t_d^{\frac{1}{12d}(q-2)(q+3)(q-1)}$$

$$+\sum_{3|d|q-1}\varphi(d)t_{\frac{d}{3}}^{\frac{2(q-1)}{d}}t_d^{\frac{1}{12d}(q^2+q-8)(q-1)}$$

$$+\sum_{2,3|d|q-1}\varphi(d)t_{\frac{d}{2}}^{\frac{q-1}{d}}t_{\frac{d}{3}}^{\frac{2(q-1)}{d}}t_d^{\frac{1}{12d}(q^2+q-14)(q-1)}$$

$$\left.+\sum_{2,3+d|q-1}\varphi(d)t_d^{\frac{1}{12d}q(q^2-1)}] + \frac{q(q-1)}{2}[\sum_{d|q+1}\varphi(d)t_d^{\frac{1}{12d}q(q^2-1)}]\right].$$

Case d)

The cycle index of G acting on the cosets of H is given by;

$$Z(G) = \left[\frac{1}{|G|} t_1^{\frac{q(q^2-1)}{12}} + (q^2-1) t_p^{\frac{q(q^2-1)}{12p}}\right.$$

$$+ \frac{q(q+1)}{2}[\sum_{2|d|q-1} \varphi(d) t_{\frac{d}{2}}^{\frac{q-1}{d}} t_d^{\frac{1}{12d}(q-2)(q+3)(q-1)}$$

$$+ \sum_{2\nmid d|q-1} \varphi(d) t_d^{\frac{1}{12d}q(q^2-1)}]$$

$$+ \frac{q(q-1)}{2}[\sum_{2|d|q+1} \varphi(d) t_{\frac{d}{2}}^{\frac{q+1}{d}} t_d^{\frac{1}{12d}(q+2)(q-3)(q+1)}$$

$$+ \sum_{3|d|q+1} \varphi(d) t_{\frac{d}{3}}^{\frac{2(q+1)}{d}} t_d^{\frac{1}{12d}((q^2-q-8)(q+1))}$$

$$+ \sum_{2,3|d|q+1} \varphi(d) t_{\frac{d}{2}}^{\frac{q+1}{d}} t_{\frac{d}{3}}^{\frac{2(q+1)}{d}} t_d^{\frac{1}{12d}(q+2)(q-3)(q+1)}$$

$$\left. + \sum_{d|q+1} \varphi(d) t_d^{\frac{1}{12d}q(q^2-1)}]\right].$$

4.5 Cycle index of G acting on the cosets of $H = A_5$

Also the cycle index of G acting on the cosets of H depends on the cases given in Section 3.5. We only prove case (a) and state a theorem on each case since the same argument is used to prove them.

4.5.1Theorem

Case a)

The cycle index of G acting on the cosets of H is given by;

$$Z(G) = \frac{1}{|G|}\Bigg[t_1^{\frac{q(q^2-1)}{60}} + (q^2-1)t_1^{\frac{q}{4}}t_p^{\frac{q(q^2-16)}{60p}}$$

$$+\frac{q(q+1)}{2}[\sum_{3|d|q-1}\varphi(d)t_{\frac{d}{3}}^{\frac{q-1}{d}}t_d^{\frac{1}{60d}(q-4)(q+5)(q-1)}$$

$$+\sum_{3,5\nmid d|q-1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}]$$

$$+\frac{q(q-1)}{2}[\sum_{5|d|q+1}\varphi(d)t_{\frac{d}{5}}^{\frac{q+1}{d}}t_d^{\frac{1}{60d}(q-4)(q+3)(q+1)}$$

$$+\sum_{3,5\nmid d|q+1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}]\Bigg].$$

Proof

From the results in Table 3.5.3 the contribution to the sum of monomial by the identity element is $t_1^{\frac{q(q^2-1)}{60}}$. The contribution by elements of τ_1 is $(q^2-1)t_1^{\frac{q}{4}}t_p^{\frac{q(q^2-16)}{60p}}$. We have two different types of monomials for elements of τ_2. These are;

i) $t_{\frac{d}{3}}^{\frac{2(q-1)}{d}}t_d^{\frac{1}{60d}(q-4)(q+5)(q-1)}$ if $3|d|q-1$.

ii) $t_d^{\frac{1}{60d}q(q^2-1)}$ if $3,5 \nmid d|q-1$.

Hence elements of τ_2 contributes;

$\frac{q(q+1)}{2}[\sum_{3|d|q-1}\varphi(d)t_{\frac{d}{3}}^{\frac{q-1}{d}}\, t_d^{\frac{1}{60d}(q-4)(q+5)(q-1)}+\sum_{3,5\nmid d|q-1}\varphi(d)t_d^{\frac{1}{60d}q\left(q^2-1\right)}]$

to the sum of the monomial. Elements of τ_0 contributes $\frac{q(q-1)}{2}[\sum_{5|d|q+1}\varphi(d)t_{\frac{d}{5}}^{\frac{q+1}{d}}\, t_d^{\frac{1}{60d}(q-4)(q+3)(q+1)}+\sum_{3,5\nmid d|q+1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}]$ to the sum of the monomials. Adding all the above contributions and dividing by $|G|$ gives the above results.■

4.5.2 Examples

The cycle index of $G = PGL(2,4)$ acting on the cosets of H is given by;

$$Z(G) = \frac{1}{60}[t_1 + 15t_1 + 20t_1 + 24t_1].$$

This is the same as the cycle index of A_5 acting on the cosets of itself as expected since $PGL(2,4) \cong A_5$.

Case b)

The cycle index of G acting on the cosets of $H$is given by;

$$Z(G)=\frac{1}{|G|}\Bigg[t_1^{\frac{q(q^2-1)}{60}}+(q^2-1)t_1^{\frac{q}{4}}t_p^{\frac{q(q^2-16)}{60p}}$$

$$+\frac{q(q+1)}{2}[\sum_{5|d|q-1}\varphi(d)t_{\frac{d}{5}}^{\frac{q-1}{d}}t_d^{\frac{(q-1)(q+4)(q-3)}{60d}}$$

$$+\sum_{3,5\nmid d|q-1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}]$$

$$+\frac{q(q-1)}{2}[\sum_{3|d|q+1}\varphi(d)t_{\frac{d}{3}}^{\frac{q+1}{d}}t_d^{\frac{(q+4)(q-5)(q+1)}{60d}}$$

$$+\sum_{3,5\nmid d|q+1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}]\Bigg].$$

Case c)

The cycle index of G acting on the cosets of $H$is given by;

$$Z(G)=\frac{1}{|G|}\Bigg[t_1^{\frac{q(q^2-1)}{60}}+(q^2-1)t_1^{\frac{q}{4}}t_p^{\frac{q(q^2-16)}{60p}}+\frac{q(q+1)}{2}$$

$$+[\sum_{3|d|q-1}\varphi(d)t_{\frac{d}{3}}^{\frac{(q-1)}{d}}t_d^{\frac{1}{60d}(q-4)(q+5)(q-1)}$$

$$+\sum_{5|d|q-1}\varphi(d)t_{\frac{d}{5}}^{\frac{q-1}{d}}t_d^{\frac{1}{60d}(q-3)(q+4)(q-1)}$$

$$+\sum_{3,5|d|q-1}\varphi(d)t_{\frac{d}{5}}^{\frac{q-1}{d}}t_{\frac{d}{3}}^{\frac{2(q-1)}{d}}t_d^{\frac{1}{60d}(q^2+q-32)(q-1)}$$

$$+\sum_{3,5\nmid d|q-1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}]+\frac{q(q-1)}{2}[\sum_{d|q+1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}]\Bigg].$$

Case d)

The cycle index of G acting on the cosets of $H$is given by;

$$Z(G)=\frac{1}{|G|}\left[t_1^{\frac{q(q^2-1)}{60}}+(q^2-1)t_1^{\frac{q}{4}}t_p^{\frac{q(q^2-16)}{60p}}\right.$$

$$+\frac{q(q+1)}{2}[\sum_{3,5\nmid d|q-1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}]$$

$$+\frac{q(q-1)}{2}[\sum_{3|d|q+1}\varphi(d)t_{\frac{d}{3}}^{\frac{(q+1)}{d}}t_d^{\frac{(q+4)(q-5)(q+1)}{60d}}$$

$$+\sum_{5|d|q+1}\varphi(d)t_{\frac{d}{5}}^{\frac{q+1}{d}}t_d^{\frac{(q+1)(q-4)(q+3)}{60d}}$$

$$+\sum_{3,5|d|q+1}\varphi(d)t_{\frac{d}{5}}^{\frac{q+1}{d}}t_{\frac{d}{3}}^{\frac{(q+1)}{d}}t_d^{\left[\frac{(q^2-q-32)(q+1)}{60d}\right]}$$

$$\left.+\sum_{d|q+1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}]\right].$$

Case e)

The cycle index of G acting on the cosets of $H$is given by;

$$Z(G) = \frac{1}{|G|}\Bigg[t_1^{\frac{q(q^2-1)}{60}} + (q^2-1)t_{\frac{d}{3}}^{\frac{q}{3}}t_p^{\frac{q}{60p}(q^2-21)}$$

$$+\frac{q(q+1)}{2}[\sum_{2|d|q-1}\varphi(d)t_{\frac{d}{2}}^{\frac{q-1}{d}}t_d^{\frac{1}{60d}(q-5)(q+6)(q-1)}$$

$$+\sum_{5|d|q-1}\varphi(d)t_{\frac{d}{5}}^{\frac{q-1}{d}}t_d^{\frac{1}{60d}(q-3)(q+4)(q-1)}$$

$$+\sum_{2,5|d|q-1}\varphi(d)t_{\frac{d}{2}}^{\frac{q-1}{d}}t_{\frac{d}{5}}^{\frac{(q-1)}{d}}t_d^{\frac{1}{60d}(q^2+q-42)(q-1)}$$

$$+\sum_{2,5\nmid d|q-1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}$$

$$+\frac{q(q-1)}{2}[\sum_{2|d|q-1}\varphi(d)t_{\frac{d}{2}}^{\frac{q+1}{d}}t_d^{\frac{(q+1)(q-6)(q+5)}{60d}}$$

$$+\sum_{d|q+1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}]\Bigg].$$

Case f)

The cycle index of G acting on the cosets of H is given by;

$$Z(G)=\frac{1}{|G|}\Bigg[t_1^{\frac{q(q^2-1)}{60}}+(q^2-1)t_{\frac{d}{3}}^{\frac{q}{3}}t_p^{\frac{q}{60p}(q^2-21)}$$

$$+\frac{q(q+1)}{2}[\sum_{2|d|q-1}\varphi(d)t_{\frac{d}{2}}^{\frac{q-1}{d}}t_d^{\frac{1}{60d}(q-5)(q+6)(q-1)}$$

$$+\sum_{2,5\nmid d|q-1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}]$$

$$+\frac{q(q-1)}{2}[\sum_{2|d|q+1}\varphi(d)t_{\frac{d}{2}}^{\frac{q+1}{d}}t_d^{\frac{(q+1)(q-6)(q+5)}{60d}}$$

$$+\sum_{5|d|q+1}\varphi(d)t_{\frac{d}{5}}^{\frac{q+1}{d}}t_d^{\frac{(q+1)(q-4)(q+3)}{60d}}$$

$$+\sum_{2,5|d|q+1}\varphi(d)t_{\frac{d}{2}}^{\frac{q+1}{d}}t_{\frac{d}{5}}^{\frac{(q+1)}{d}}t_d^{\frac{(q+1)(q^2-q-42)}{60d}}$$

$$+\sum_{d|q+1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}]\Bigg].$$

Case g)

The cycle index of G acting on the cosets of $H$is given by;

$$Z(G) = \frac{1}{|G|}\Bigg[t_1^{\frac{q(q^2-1)}{60}} + (q^2-1)t_{\frac{d}{5}}^{\frac{q}{5}} t_p^{\frac{q}{60p}(q^2-25)}$$

$$+\frac{q(q+1)}{2}[\sum_{2|d|q-1} \varphi(d) t_{\frac{d}{2}}^{\frac{q-1}{d}} t_d^{\frac{1}{60d}(q-5)(q+6)(q-1)}$$

$$+\sum_{3|d|q-1} \varphi(d) t_{\frac{d}{3}}^{\frac{(q-1)}{d}} t_d^{\frac{1}{60d}(q-4)(q+5)(q-1)}$$

$$+\sum_{2,3|d|q-1} \varphi(d) t_{\frac{d}{2}}^{\frac{q-1}{d}} t_{\frac{d}{3}}^{\frac{(q-1)}{d}} t_d^{\frac{1}{60d}(q^2+q-50)(q-1)}$$

$$+\sum_{2,3\nmid d|q-1} \varphi(d) t_d^{\frac{1}{60d}q(q^2-1)}]$$

$$+\frac{q(q-1)}{2}[\sum_{2|d|q+1} \varphi(d) t_{\frac{d}{2}}^{\frac{q+1}{d}} t_d^{\frac{(q+1)(q-6)(q+5)}{60d}}$$

$$+\sum_{d|q+1} \varphi(d) t_d^{\frac{1}{60d}q(q^2-1)}]\Bigg].$$

Case h)

The cycle index of G acting on the cosets of H is given by;

$$Z(G)=\frac{1}{|G|}\Bigg[t_1^{\frac{q(q^2-1)}{60}}+(q^2-1)t_{\frac{d}{5}}^{\frac{q}{5}}t_p^{\frac{q}{60p}(q^2-25)}$$

$$+\frac{q(q+1)}{2}[\sum_{2|d|q-1}\varphi(d)t_{\frac{d}{2}}^{\frac{q-1}{d}}t_d^{\frac{1}{60d}(q-5)(q+6)(q-1)}$$

$$+\sum_{2\nmid d|q-1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}]$$

$$+\frac{q(q-1)}{2}[\sum_{2|d|q+1}\varphi(d)t_{\frac{d}{2}}^{\frac{q+1}{d}}t_d^{\frac{(q+1)(q-6)(q+5)}{60d}}$$

$$+\sum_{3|d|q+1}\varphi(d)t_{\frac{d}{3}}^{\frac{(q+1)}{d}}t_d^{\frac{(q+4)(q-5)(q+1)}{60d}}$$

$$+\sum_{2,3|d|q+1}\varphi(d)t_{\frac{d}{2}}^{\frac{q+1}{d}}t_{\frac{d}{3}}^{\frac{(q+1)}{d}}t_d^{\frac{(q+1)(q^2-q-50)}{60d}}+\sum_{d|q+1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}]\Bigg]$$

Case i)

The cycle index of G acting on the cosets of H is given by;

$$
\begin{aligned}
Z(G) \quad &= \frac{1}{|G|}\left[t_1^{\frac{q(q^2-1)}{60}} + (q^2-1)t_p^{\frac{q}{60p}(q^2-1)} \right.\\
&+ \frac{q(q+1)}{2}[\sum_{2|d|q-1} \varphi(d) t_{\frac{d}{2}}^{\frac{q-1}{d}} t_d^{\frac{1}{60d}(q-5)(q+6)(q-1)}\\
&+ \sum_{2\nmid d|q-1} \varphi(d) t_d^{\frac{1}{60d}q(q^2-1)}]\\
&+ \frac{q(q-1)}{2}[\sum_{2|d|q+1} \varphi(d) t_{\frac{d}{2}}^{\frac{q+1}{d}} t_d^{\frac{(q+1)(q-6)(q+5)}{60d}}\\
&+ \sum_{3|d|q+1} \varphi(d) t_{\frac{d}{3}}^{\frac{(q+1)}{d}} t_d^{\frac{(q+4)(q-5)(q+1)}{60d}}\\
&+ \sum_{2,3|d|q+1} \varphi(d) t_{\frac{d}{2}}^{\frac{q+1}{d}} t_{\frac{d}{3}}^{\frac{(q+1)}{d}} t_d^{\frac{(q+1)(q^2-q-50)}{60d}}\\
&+ \sum_{5|d|q+1} \varphi(d) t_{\frac{d}{5}}^{\frac{q+1}{d}} t_d^{\frac{(q+1)(q-4)(q+3)}{60d}}\\
&+ \sum_{2,5|d|q+1} \varphi(d) t_{\frac{d}{2}}^{\frac{q+1}{d}} t_{\frac{d}{5}}^{\frac{(q+1)}{d}} t_d^{\frac{(q+1)(q^2-q-42)}{60d}}\\
&+ \sum_{3,5|d|q+1} \varphi(d) t_{\frac{d}{5}}^{\frac{q+1}{d}} t_{\frac{d}{3}}^{\frac{(q+1)}{d}} t_d^{\left[\frac{(q^2-q-32)(q+1)}{60d}\right]}\\
&+ \sum_{2,3,5|d|q+1} \varphi(d) t_{\frac{d}{2}}^{\frac{q+1}{d}} t_{\frac{d}{3}}^{\frac{(q+1)}{d}} t_{\frac{d}{5}}^{\frac{(q+1)}{d}} t_d^{\frac{(q+1)(q^2-q-62)}{60d}}\\
&\left. + \sum_{d|q+1} \varphi(d) t_d^{\frac{1}{60d}q(q^2-1)}] \right]
\end{aligned}
$$

Case j)

The cycle index of G acting on the cosets of H is given by;

$$Z(G)=\frac{1}{|G|}\Bigg[t_1^{\frac{q(q^2-1)}{60}}+(q^2-1)t_p^{\frac{q}{60p}(q^2-1)}$$

$$+\frac{q(q+1)}{2}[\sum_{2|d|q-1}\varphi(d)t_{\frac{d}{2}}^{\frac{q-1}{d}}t_d^{\frac{1}{60d}(q-5)(q+6)(q-1)}$$

$$+\sum_{5|d|q-1}\varphi(d)t_{\frac{d}{5}}^{\frac{q-1}{d}}t_d^{\frac{1}{60d}(q-3)(q+4)(q-1)}$$

$$+\sum_{2,5|d|q-1}\varphi(d)t_{\frac{d}{2}}^{\frac{q-1}{d}}t_{\frac{d}{5}}^{\frac{(q-1)}{d}}t_d^{\frac{1}{60d}(q^2+q-42)(q-1)}$$

$$+\sum_{2\nmid d|q-1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}]$$

$$+\frac{q(q-1)}{2}[\sum_{2|d|q+1}\varphi(d)t_{\frac{d}{2}}^{\frac{q+1}{d}}t_d^{\frac{(q+1)(q-6)(q+5)}{60d}}$$

$$+\sum_{3|d|q+1}\varphi(d)t_{\frac{d}{3}}^{\frac{(q+1)}{d}}t_d^{\frac{(q+4)(q-5)(q+1)}{60d}}$$

$$+\sum_{2,3|d|q+1}\varphi(d)t_{\frac{d}{2}}^{\frac{q+1}{d}}t_{\frac{d}{3}}^{\frac{(q+1)}{d}}t_d^{\frac{(q+1)(q^2-q-50)}{60d}}$$

$$+\sum_{d|q+1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}]\Bigg].$$

Case k)

The cycle index of Gacting on the cosets of H is given by;

$$Z(G)=\frac{1}{|G|}\Bigg[t_1^{\frac{q(q^2-1)}{60}}+(q^2-1)t_p^{\frac{q(q^2-1)}{60p}}$$

$$+\frac{q(q+1)}{2}[\sum_{2|d|q-1}\varphi(d)t_{\frac{d}{2}}^{\frac{q-1}{d}}t_d^{\frac{1}{60d}(q-5)(q+6)(q-1)}$$

$$+\sum_{3|d|q-1}\varphi(d)t_{\frac{d}{3}}^{\frac{(q-1)}{d}}t_d^{\frac{1}{60d}(q-4)(q+5)(q-1)}$$

$$+\sum_{5|d|q-1}\varphi(d)t_{\frac{d}{5}}^{\frac{q-1}{d}}t_d^{\frac{1}{60d}(q-3)(q+4)(q-1)}$$

$$+\sum_{2,3|d|q-1}\varphi(d)t_{\frac{d}{2}}^{\frac{q-1}{d}}t_{\frac{d}{3}}^{\frac{(q-1)}{d}}t_d^{\frac{1}{60d}(q^2+q-50)(q-1)}$$

$$+\sum_{2,5|d|q-1}\varphi(d)t_{\frac{d}{2}}^{\frac{q-1}{d}}t_{\frac{d}{5}}^{\frac{(q-1)}{d}}t_d^{\frac{1}{60d}(q^2+q-42)(q-1)}$$

$$+\sum_{3,5|d|q-1}\varphi(d)t_{\frac{d}{5}}^{\frac{q-1}{d}}t_{\frac{d}{3}}^{\frac{2(q-1)}{d}}t_d^{\frac{1}{60d}(q^2+q-32)(q-1)}$$

$$+\sum_{2,3,5|d|q-1}\varphi(d)t_{\frac{d}{2}}^{\frac{q-1}{d}}t_{\frac{d}{3}}^{\frac{2(q-1)}{d}}t_{\frac{d}{5}}^{\frac{(q-1)}{d}}t_d^{\frac{1}{60d}(q^2+q-62)(q-1)}$$

$$+\sum_{2,3,5+d|q-1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}]$$

$$+\frac{q(q-1)}{2}[\sum_{d|q+1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}]\Bigg]$$

Case l

The cycle index of Gacting on the cosets of $H$is given by;

$$Z(G) = \frac{1}{|G|}\left[t_1^{\frac{q(q^2-1)}{60}} + (q^2-1)t_p^{\frac{q(q^2-1)}{60p}}\right.$$

$$+\frac{q(q+1)}{2}[\sum_{2|d|q-1}\varphi(d)t_{\frac{d}{2}}^{\frac{q-1}{d}}t_d^{\frac{1}{60d}(q-5)(q+6)(q-1)}$$

$$+\sum_{3|d|q-1}\varphi(d)t_{\frac{d}{3}}^{\frac{(q-1)}{d}}t_d^{\frac{1}{60d}(q-4)(q+5)(q-1)}$$

$$+\sum_{2,3|d|q-1}\varphi(d)t_{\frac{d}{2}}^{\frac{q-1}{d}}t_{\frac{d}{3}}^{\frac{(q-1)}{d}}t_d^{\frac{1}{60d}(q^2+q-50)(q-1)}$$

$$+\sum_{2,3,\nmid d|q-1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}$$

$$+\frac{q(q-1)}{2}[\sum_{2|d|q+1}\varphi(d)t_{\frac{d}{2}}^{\frac{q+1}{d}}t_d^{\frac{(q+1)(q-6)(q+5)}{60d}}$$

$$+\sum_{5|d|q+1}\varphi(d)t_{\frac{d}{5}}^{\frac{q+1}{d}}t_d^{\frac{(q+1)(q-4)(q+3)}{60d}}$$

$$+\sum_{2,5|d|q+1}\varphi(d)t_{\frac{d}{2}}^{\frac{q+1}{d}}t_{\frac{d}{5}}^{\frac{(q+1)}{d}}t_d^{\frac{(q+1)(q^2-q-42)}{60d}}$$

$$\left.+\sum_{d|q+1}\varphi(d)t_d^{\frac{1}{60d}q(q^2-1)}]\right].$$

4.6 Cycle index of G acting on the cosets of $H = D_{2(q-1)}$

The cycle index of G acting on the cosets of H depends on the cases given in section 3.6.

Case a)

4.6.1 Theorem

The Cycle index of G acting on the cosets of H when q is odd is given by;

$$Z(G) = \frac{1}{|G|}\Bigg[t_1^{|G|/|H|} + (q^2-1)t_p^{p^{f-1}\left(\frac{q+1}{2}\right)}$$

$$+\frac{q(q+1)}{2}\sum_{2|d|q-1}\varphi(d)t_1 t_{\frac{d}{2}}^{\frac{q-1}{d}} t_d^{\frac{1}{2d}(q^2-1)}$$

$$+\frac{q(q+1)}{2}\sum_{d|q-1}\varphi(d)t_1 t_d^{\frac{1}{2d}(q-1)(q+2)}$$

$$+\frac{q(q-1)}{2}\sum_{2|d|q+1}\varphi(d)t_{\frac{d}{2}}^{\frac{q+1}{d}} t_d^{\frac{1}{2d}(q^2-1)}$$

$$+\frac{q(q-1)}{2}\sum_{d|q+1}\varphi(d)t_d^{\frac{q(q+1)}{2d}}\Bigg].$$

Proof

Using the results from Table 3.6.2 and argument similar to those in Theorem 4.1.2 the theorem follows.■

4.6.2 Example

The cycle index of $PGL(2,3)$ acting on the cosets of H is given by;

$$Z(G) = \frac{1}{24}[t_1^6 + 8t_3^2 + 6t_1^2t_2^2 + 3t_1^2t_2^2 + 6t_2t_4]$$

Case b)

4.6.3 Theorem

The Cycle index of G acting on the cosets of H when q is even is given by;

$$(G) = \frac{1}{|G|}\left[t_1^{|G|/|H|} + (q^2-1)t_1^{2^{f-1}}t_p^{4^{f-1}} + \frac{q(q+1)}{2}\sum_{2|d|q-1}\varphi(d)t_1t_d^{\frac{1}{2d}(q^2-1)} + \right.$$

$$\left. + \frac{q(q-1)}{2}\sum_{d|q+1}\varphi(d)t_d^{\frac{q(q+1)}{2d}}\right]$$

CHAPTER FIVE

RANKS AND SUBDEGREES OF $G = PGL(2, q)$ ON THE COSETS OF ITS SUBGROUPS

In this chapter we compute the ranks and the subdegrees of the permutation representations of G on the cosets of its subgroups. This chapter has two main sections. In Section 5.1 we find the ranks of G on the cosets C_{q-1}, C_{q+1}, P_q, A_4, A_5 and $D_{2(q-1)}$ using the results in Chapter 3. In Section 5.2 we determine the subdegrees of G on the cosets of C_{q-1}, C_{q+1}, P_q, A_4,A_5 and $D_{2(q-1)}$ using the table of marks. We also confirm the ranks computed in Section 5.1.

5.1 Ranks G on the cosets of its subgroups

In this section we compute the ranks of G on the cosets of its subgroups using Theorem 1.1.11 and the results in Chapter 3.

5.1.1 Rank of G on the cosets of $H = C_{q-1}$

By using Theorem 1.1.11, we calculate the rank (*r)* of *G* as follows;

From the result in Table 3.1.1, elements of *H* have fixed points as follows;

The identity fixes $q(q + 1)$ cosets, the remaining $q - 2$ element each fixes 2 cosets.

Thus we have;

$$r = \frac{1}{q-1}\{q(q+1) + 2(q-2)\}$$

$$= \frac{1}{q-1}\{(q-1)(q+4)\}$$

$$= (q+4).$$

5.1.2 Rank of G on the cosets of $H = C_{q+1}$

From the results in Table 3.2.1 above and using Cauchy Frobenius Lemma (Theorem 1.1.11) we calculate the rank (r) as follows;

The identity fixes $q(q-1)$ cosets and the remaining q elements each fixes 2 cosets.

Thus we have;

$$r = \frac{1}{q+1}[q(q-1) + 2q]$$

$$= \frac{1}{q+1}[q^2 + q]$$

$$= q.$$

5.1.3 Rank of G on the cosets of $H = P_q$

From the results in Table 3.3.1 we calculate the rank r as follows;

The identity fixes $q^2 - 1$ cosets and the remaining $q - 1$ elements each fixes $q - 1$ cosets.

Thus we have;

$$r = \frac{1}{q}[(q^2 - 1) + (q-1)(q-1)]$$

$$= \frac{1}{q}[(q^2 - 1 + q^2 - 2q + 1)]$$

$$= 2(q-1).$$

5.1.4 Ranks of G on the cosets of $H = A_4$

In computing the ranks (r) of G on the cosets of *H* we need to look at all the four cases in Section 3.4.

Case a)

From the results in Table 3.4.2 we establish that, elements of *H* have fixed points as follows;

The identity fixes $\frac{q(q^2-1)}{12}$ cosets. We also have 3 elements of order two each fixing $\frac{q}{4}$ cosets and 8 elements of order three each fixing $\frac{2}{3}(q-1)$ cosets. Hence by Theorem 1.1.11 we have;

$$r = \frac{1}{12}\left[\frac{q(q^2-1)}{12} + 3\frac{q}{4} + 8(\frac{2}{3}(q-1))\right]$$

$$= \frac{1}{12}\left[\frac{q^3 - q + 9q + 64q - 64}{12}\right]$$

$$= \frac{1}{144}[q^3 + 72q - 64].$$

NB: We only give results for the other three cases since the argument used is the same as in case a) above.

Case b)

$$r = \frac{1}{12}\left[\frac{q(q^2-1)}{12} + 3\left(\frac{q-1}{2}\right) + 8\left(\frac{2}{3}q\right)\right]$$

$$= \frac{1}{144}[q^3 + 81q - 18].$$

Case c)

$$r = \frac{1}{12}\left[\frac{q(q^2-1)}{12} + 3\left(\frac{q-1}{2}\right) + 8\left(\frac{2}{3}(q-1)\right)\right]$$

$$= \frac{1}{144}[q^3 + 81q - 82].$$

Case d)

$$r = \frac{1}{12}\left[\frac{q(q^2-1)}{12} + 3\left(\frac{q+1}{2}\right) + 8\left(\frac{2}{3}(q+1)\right)\right]$$

$$= \frac{1}{144}[q^3 + 81q + 82].$$

5.1.5 Ranks of G on the cosets of $H = A_5$

To find the ranks of G on the cosets of H we need to look at all the cases in Section 3.5;

Case a)

From the results in Table 3.5.2 we observe that the identity fixes $\frac{q(q^2-1)}{60}$ cosets. The 15 elements of order 2 each fixes $\frac{q}{4}$ cosets. The twenty elements of order 3 each fixes $\frac{1}{3}(q-1)$ cosets. Finally the 24 elements of order five each fixes $\frac{1}{5}(q+1)$ cosets. Now applying the Theorem 1.1.11 we obtain;

$$r = \frac{1}{60}\left[\frac{q(q^2-1)}{60} + 15\frac{q}{4} + 20\frac{1}{3}(q-1) + 24\frac{1}{5}(q+1)\right]$$

$$= \frac{q^3 + 912q - 112}{3600}.$$

NB: Since the approach of finding the ranks for the remaining cases are the same as case a) we only give results.

Case b)

$$r = \frac{1}{60}\left[\frac{q(q^2-1)}{60} + 15\frac{q}{4} + 20\frac{1}{3}(q+1) + 24\frac{1}{5}(q-1)\right]$$

$$= \frac{q^3+912q+112}{3600}.$$

Case c)

$$r = \frac{1}{60}\left[\frac{q(q^2-1)}{60} + 15\frac{q}{4} + 20\frac{1}{3}(q+1) + 24\frac{1}{5}(q-1)\right]$$

$$= \frac{q^3+912q+114}{3600}.$$

Case d)

$$r = \frac{1}{60}\left[\frac{q(q^2-1)}{60} + 15\frac{q}{4} + 20\frac{1}{3}(q-1) + 24\frac{1}{5}(q-1)\right]$$

$$= \frac{q^3+912q-688}{3600}.$$

Case e)

$$r = \frac{1}{60}\left[\frac{q(q^2-1)}{60} + 15\left(\frac{q-1}{2}\right) + 20\frac{1}{3}q + 24\frac{1}{5}(q-1)\right]$$

$$= \frac{q^3+1137q-738}{3600}.$$

Case f)

$$r = \frac{1}{60}\left[\frac{q(q^2-1)}{60} + 15\left(\frac{q+1}{2}\right) + 20\frac{1}{3}q + 24\frac{1}{5}(q+1)\right]$$

$$= \frac{q^3+1137q+738}{3600}.$$

Case g)

$$r = \frac{1}{60}\left[\frac{q(q^2-1)}{60} + 15\left(\frac{q-1}{2}\right) + 20\frac{1}{3}(q-1) + 24\frac{1}{5}q\right]$$

$$= \frac{q^3 + 1137q - 850.}{3600}$$

Case h)

$$r = \frac{1}{60}\left[\frac{q(q^2-1)}{60} + 15\left(\frac{q+1}{2}\right) + 20\frac{1}{3}(q+1) + 24\frac{1}{5}q\right]$$

$$= \frac{q^3 + 1137q + 850}{3600}.$$

Case i)

$$r = \frac{1}{60}\left[\frac{q(q^2-1)}{60} + 15\left(\frac{q-1}{2}\right) + 20\frac{1}{3}(q-1) + 24\frac{1}{5}(q-1)\right]$$

$$= \frac{q^3 + 1137q - 1138}{3600}.$$

Case j)

$$r = \frac{1}{60}\left[\frac{q(q^2-1)}{60} + 15\left(\frac{q-1}{2}\right) + 20\frac{1}{3}(q+1) + 24\frac{1}{5}(q-1)\right]$$

$$= \frac{q^3 + 1137q - 338}{3600}.$$

Case k)

$$r = \frac{1}{60}\left[\frac{q(q^2-1)}{60} + 15\left(\frac{q+1}{2}\right) + 20\frac{1}{3}(q+1) + 24\frac{1}{5}(q+1)\right]$$

$$= \frac{q^3 + 1137q + 1138}{3600}.$$

Case l)

$$r = \frac{1}{60}\left[\frac{q(q^2-1)}{60} + 15\left(\frac{q-1}{2}\right) + 20\frac{1}{3}(q-1) + 24\frac{1}{5}(q+1)\right]$$

$$= \frac{q^3 + 1137q - 562}{3600}.$$

5.1.6 Ranks of G on the cosets of $H = D_{2(q-1)}$

We have two cases to consider

a) When q is odd

b) When q is even

Case a)

From the results in Table 3.6.1 we establish that, elements of H have fixed points as follows:

The identity fixes $\frac{q(q+1)}{2}$ cosets. The q involutions each fixes $\frac{q+1}{2}$. Finally the remaining $q-3$ elements of order greater than two each fixes a single coset. Now applying Theorem 1.1.11 we have;

$$r = \frac{1}{2(q-1)}\left[\frac{q(q+1)}{2} + \frac{q(q+1)}{2} + (q-3))\right]$$

$$= \left[\frac{q+3}{2}\right].$$

Case b)

Using the same argument as case i) above we establish that the rank is given by;

$$r = \frac{1}{2(q-1)}\left(\frac{q(q+1)}{2} + (q-1)2^{f-1} + q - 2\right)$$

$$= \frac{q+2}{2}$$

5.2 Subdegrees G on the cosets of its subgroups

In this section we compute the subdegrees of G using the table of marks. We also confirm the ranks computed in the previous section.

5.2.1 Subdegrees of G on the cosets of $H = C_{q-1}$

Before finding the Subdegrees of G on the cosets of H we first need to give some theorems which will be used in this section and other sections to follow.

5.2.1.1 Lemma

Let C_d (d coprime to p) be a cyclic subgroup of order d, then;

$$N_G(C_d) = \begin{cases} D_{q\pm1,} & p\ odd \\ D_{2(q\pm1)} & p = 2 \end{cases}$$

$\pm$ sign as $d|q \pm 1$. (Dickson, 1901)

5.2.1.2 Lemma

Let C_p be a cyclic subgroup of order p in G, then

$$N_G(C_p) = \begin{cases} \frac{1}{2}q(p-1)\ p\ odd, f\ odd \\ q(p-1) p\ odd\ f\ even \\ q \qquad\qquad p = 2 \end{cases}$$

(Dickson, 1901)

5.2.1.3 Lemma

Let $d > 3$ be a divisor of $(q \pm 1)$ and δ be the quotient, then

$$N_G(D_{2d}) = \begin{cases} D_{2d,} & if\ \delta\ is\ odd \\ D_{4d} & if\ \delta\ is\ even \end{cases}$$

(Dickson, 1901)

Since H is abelian, each of its subgroups is normal. Suppose H has s subgroups say, $H_1 = 1, H_2, H_3, \dots \dots \dots \dots, H_s = H$ with $i|q - 1$ and,

$(i = 1,2, \dots, s - 1)$.

Now using the method proposed by Ivanov et al.(1983) the table of marks of H can be computed as follows;

Table 5.2.1.1:Table of marks of $\boldsymbol{H = C_{q-1}}$

	H_1	H_2	$\dots \dots \dots \dots$	H_s
$H(/H_1)$	$q - 1$			
$H(/H_2)$	m_{21}	m_{22}		
….	……….		.	
….	… . …….			
$H(/H_s)$	1	1	$\dots \dots \dots$	1

After computing the table of marks of H, we now proceed to find

$m(F) = m(F, H, G)$, where F is a representative of a conjugacy class in H and $F \leq H$. The value of $m(F)$ is obtained using Lemma 5.2.1.1 and the method proposed by Ivanov et al. (1983). The values of $m(F)$ are displayed in Table 5.2.1.2 below.

Table 5.2.1.2: The mark of *F* where $\boldsymbol{F \leq H = C_{q-1}}$

F	$H_1 = I$	H_2	H_3	$H_{4,}$	$\cdots$	H_{n-1}	$H_n = G$
$M(F)$	$q(q+1)$	2	2	2	$\cdots$	2	2

Let $Q = (Q_{1,}Q_{2,}Q_{3,} \dots \dots .., Q_{S-1,}\ Q_S)$ denote the number of suborbits $\Delta_{\boldsymbol{j}}$. Then by Theorem 1.1.19 and using Table 5.2.1.2 and Table 5.2.1.1 we obtain the following system of equations;

$$(q-1)Q_1 + m_{21}Q_2 + \cdots + m_{s-11}Q_{s-1} + Q_s = q(q+1)$$

$$m_{22}Q_2 + \cdots + m_{s-12}Q_{s-1} + Q_s = 2$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$m_{s-1s-1}Q_{s-1} + Q_s = 2$$

$$Q_s = 2.$$

Solving the above system of equations we obtain $Q = (q+2, 0,0, \dots \dots .,2)$.

Hence the subdegrees of *G* on the cosets of *H* are shown in Table 5.2.1.3 below

Table 5.2.1.3: Subdegrees of *G* on the cosets of $\boldsymbol{C_{q-1}}$

Suborbit Length	1	$q-1$
No. of Suborbits	2	$q+2$

Therefore the rank (r) is given by;

$$r = 2 + q + 2 = q + 4$$

5.2.2 Subdegrees of $\boldsymbol{G}$ on the cosets of $\boldsymbol{H = C_{q+1}}$

Since H is cyclic thus it is abelian, so each of its subgroups is normal. If H has s subgroups say, $H_1 = 1, H_2, H_3, \dots\dots\dots\dots, H_s = H$ with $i|q+1$ and $(i = 1, 2, \dots, s-1)$.

Again using the method proposed by Ivanov et al.(1983) the table of marks of H can be as shown below;

Table 5.2.2.1: Table of marks of $\boldsymbol{H = C_{q+1}}$

	H_1	H_2	$\dots\dots\dots\dots$	H_s
$H(/H_1)$	$q+1$			
$H(/H_2)$	m_{21}	m_{22}		
….	…	….	…	
….	…	….	……….	
$H(/H_s)$	1	1	$\dots\dots\dots$	1

Using Lemma 5.2.1.1 and Lemma 5.2.1.2 we find $m(F) = (q(q-1), 2,2, \dots, 2)$, s −tuples. Let $Q = (Q_1, Q_2, Q_3, \dots\dots\dots, Q_s)$. Then using Table 5.2.2.1 and Theorem 1.1.19 we form the following systems of equations;

$$(q+1)Q_1 + m_{21}Q_2 + \cdots + m_{s-11}Q_{s-1} + Q_s = q(q-1)$$

$$m_{22}Q_2 + \cdots + m_{s-12}Q_{s-1} + Q_s = 2$$

$$\dots \dots \dots \dots \dots \dots \dots \dots$$

$$m_{s-1s-1}Q_{s-1} + Q_s = 2$$

$$Q_s = 2$$

Solving the above system of equations we obtain $Q = (q-2, 0,0, \dots \dots ,0,2)$.

Hence the subdegrees of *G* on the cosets of *H* are shown in Table 5.2.2.2 below

Table 5.2.2.2: Subdegrees of *G* on the cosets of $\boldsymbol{H = C_{q+1}}$

Suborbit Length	1	$q+1$
No. of Suborbits	2	$q-2$

Therefore the rank (r) is given by;

$$r = 2 + q - 2 = q$$

5.2.3 Subdegrees of $\boldsymbol{G}$ on the cosets of $\boldsymbol{H = P_q}$

Suppose H has n subgroups say $H_1 = I,\ H_{2,}\, H_{3,}\, H_{4,} \dots \dots \dots \dots . H_n = H$. These subgroups are of order $1, p, p^2, p^3, \dots \dots \dots, p^n$ respectively since *H* is a *p*-group. Also all these subgroups are normal in *H*. So the table of marks of *H* omitting the zeros above the main diagonal is as shown in Table 5.2.3.1 below.

Table 5.2.3.1:Table of marks of $\boldsymbol{H = P_q}$

	$H_1 = I$ H_2 H_3 $H_{4,}$ … … … … … … H_{n-1} $H_n = H$.
$H(/H_1)$	$q = p^n$
$H(/H_2)$	p^{n-1} p^{n-1}
$H(/H_3)$	p^{n-2} p^{n-2} p^{n-2}
$H(/H_4)$	p^{n-3} p^{n-3} p^{n-3} p^{n-3}
……..	………………………………..
……..	………………………………………
$H(/H_{n-1})$	p p p p … … p
$H(/H_n)$	1 1 … … … . . 1

After computing the table of marks of H, we now proceed to find

$m(F) = m(F, H, G)$, where F is a representative of a conjugacy class in H and $F \leq H$. The values of $m(F)$ are displayed in Table 5.2.3.2 below.

Table 5.2.3.2:The mark of F, where $\boldsymbol{F \leq H = P_q}$

F	$H_1 = I$	H_2	H_3	$H_{4,}$	……	H_{n-1}	$H_n = H$
$M(F)$	$q^2 - 1$	$q - 1$	$q - 1$	$q - 1$	…	$q - 1$	$q - 1$

Let $Q = (Q_{1,}Q_{2,}Q_{3,} \dots \dots . . , Q_{n-1,} Q_n)$ denote the number of suborbits $\Delta_{\boldsymbol{j}}$. Then by Theorem 1.1.19 and using Table 5.2.4.1 and Table 5.2.3.2 we obtain the following system of equations;

$$p^nQ_1 + p^{n-1}Q_2 + P^{n-2}Q_3 + P^{n-3}Q_4 + \cdots \ldots\ldots\ldots.. + pQ_{n-1} + Q_n = q^2 - 1$$
$$p^{n-1}Q_2 + P^{n-2}Q_3 + P^{n-3}Q_4 + \cdots \ldots\ldots\ldots.. + pQ_{n-1} + Q_n = q - 1$$

$$P^{n-2}Q_3 + P^{n-3}Q_4 + \cdots \ldots\ldots\ldots.. + pQ_{n-1} + Q_n = q - 1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$pQ_{n-1} + Q_n = q - 1$$
$$Q_n = q - 1$$

Solving the above systems of equation we obtain $Q = (q - 1, 0, 0, \ldots..,0, q - 1)$.
Hence the subdegrees of *G* on the cosets of *H* are shown in Table 5.2.3.3 below.

Table 5.2.3.3: Subdegrees of $\boldsymbol{G}$ on the cosets of $\boldsymbol{H = P_q}$

Suborbit Length	1	q
No. of Suborbits	$q - 1$	$q - 1$

Therefore the rank (r) is given by;

$$r = (q - 1) + (q - 1) = 2(q - 1)$$

5.2.4 Subdegrees of $\boldsymbol{G}$ on the cosets of $\boldsymbol{H = A_4}$

The following are all the conjugacy classes of subgroups of H

H_1 – Identity subgroup

H_2 – 3 conjugate cyclic subgroups of order 2, C_2

H_3 – 4 conjugate cyclic subgroups of order 3, C_3

H_4 – A normal subgroup of order 4 isomorphic to $C_2 \times C_2$

H_5 – A_4

(Kamuti, 1992, P.90)

The corresponding table of marks of H omitting zeros above the main diagonal is as shown in Table 5.2.4.1 below

Table 5.2.4.1: Table of marks of $\boldsymbol{H = A_4}$

	$H_1 = I$	H_2	H_3	H_4,	$H_5 = H$
$H(/H_1)$	12				
$H(/H_2)$	6	2			
$H(/H_3)$	4	0	1		
$H(/H_4)$	3	3	0	3	
$H(/H_5)$	1	1	1	1	1

After computing the table of marks of H we proceed to find $m(F) = m(F, H, G)$, where F is a representative of a conjugacy class in H and $F \leq H$.

Our computations will be carried out under the following four cases;

a) $p = 2, \quad q \equiv 1 \, mod \, 3$

b) $p = 3$

c) $p > 3, \quad q \equiv 1 \, mod \, 3$

d) $p > 3, \quad q \equiv -1 \, mod \, 3$

Case a)

To find $m(F)$ we first need to determine $|N_G(F)|$ and $|N_H(F)|$. We obtain $|N_G(F)|$ using Lemma 5.2.1.1, Lemma 5.2.1.2 and Lemma 5.2.1.3 . We compute $|N_H(F)|$ using Lemma 1.2.3.2. After obtaining the above results we now proceed to compute $m(F)$. Table 5.2.4.2 below gives the values of $m(F)$.

Table 5.2.4.2: The mark of F, where $\boldsymbol{F \leq H = A_4}$ when $\boldsymbol{p = 2, q \equiv 1 \bmod 3}$

F	$\|N_G(F)\|$	$\|N_H(F)\|$.	$m(F)$
$H_1 = \mathrm{I}$	$q(q^2-1)$	12	$\frac{q(q^2-1)}{12}$
$H_2 = \mathrm{C}_2$	q	4	$\frac{q}{4}$
$H_3 = \mathrm{C}_3$	$2(q-1)$	3	$\frac{2(q-1)}{3}$
$H_4 = \mathrm{V}_4$	24	12	2
$H_5 = \mathrm{A}_4$	24	12	2

Let $Q = (Q_{1,}Q_{2,}Q_{3,}Q_4, Q_5)$ denote the number of suborbits $\Delta_{\boldsymbol{j}}$. Then by Theorem 1.1.19 and using Table 5.2.4.1 and Table 5.2.4.2 we obtain the following system of equations;

$$12Q_1 + 6Q_2 + 4Q_3 + 3Q_4 + Q_5 = \frac{q(q^2-1)}{12}$$

$$2Q_2 + 0Q_3 + 3Q_4 + Q_5 = \frac{q}{4}$$

$$Q_3 + 0Q_4 + Q_5 = \frac{2(q-1)}{3}$$

$$3Q_4 + Q_5 = 2$$

$$Q_5 = 2.$$

Solving the above system of equations we obtain;

$$Q = (Q_5 = 2, Q_4 = 0, Q_3 = \frac{2q-8}{3}, Q_2 = \frac{q-8}{8}, Q_1 = \frac{q^3 - 42q + 176}{144}).$$

The rank of *G* is given by;

$$r = 2 + \frac{2q-8}{3} + \frac{q-8}{8} + \frac{q^3 - 42q + 176}{144}$$

$$= \frac{1}{144}[q^3 + 72q - 64].$$

Hence the subdegrees of G on the cosets of H are as shown in Table 5.2.4.3

Table 5.2.4.3: Subdegrees of *G* on the cosets of $\boldsymbol{A_4}$ when $\boldsymbol{p = 2, q \equiv 1 \bmod 3}$

Suborbit Length	1	4	6	12
No. of Suborbits	2	$\frac{2q-8}{3}$	$\frac{q-8}{8}$	$\frac{q^3-42q+176}{144}$

NB: Since the method of finding the subdegrees in the remaining cases is the same as case a), we are going to quote the results only.

Case b)

$Q = (Q_5 = 2,\ Q_4 = 0,\ Q_3 = \frac{2q-6}{3},\ Q_2 = \frac{q-5}{4},\ Q_1 = \frac{q^3-51q+162}{144})$

Hence the rank(r) is,

$$r = 2 + \frac{2q-6}{3} + \frac{q-5}{4} + \frac{q^3 - 51q + 162}{144}$$

$$= \frac{q^3 + 81q - 18}{144}.$$

Therefore the subdegrees of G on the cosets of H are as shown in Table 5.2.4.4.

Table 5.2.4.4: Subdegrees of G on the cosets of $\boldsymbol{H = A_4}$ where $\boldsymbol{p = 3}$

Suborbit Length	1	4	6	12
No. of Suborbits	2	$\frac{q-6}{3}$	$\frac{q-5}{4}$	$\frac{q^3-35q+162}{144}$

Case c)

$Q = (Q_5 = 2, Q_4 = 0,\ Q_3 = \frac{2q-8}{3}\ ,\ Q_2 = \frac{q-5}{4},\ Q_1 = \frac{q^3-51q+194}{144})$

The rank is given by;

$$r = 2 + \frac{2q-8}{3} + \frac{q-5}{4} + \frac{q^3-51q+194}{144}$$
$$= \frac{1}{144}[q^3 + 81q - 82].$$

Hence the subdegrees of G on the cosets of H are as shown in Table 5.2.4.5

Table 5.2.4.5: Subdegrees of G on the cosets of $\boldsymbol{H = A_4}$ when $\boldsymbol{p} > 3\ \boldsymbol{q \equiv 1 mod 3}$

Suborbit Length	1	4	6	12
No. of Suborbits	2	$\frac{q-7}{3}$	$\frac{q-6}{4}$	$\frac{q^3-35q+196}{144}$

Case d)

$Q = (Q_5 = 2, Q_4 = 0,\ Q_3 = \frac{2q-4}{3}\ ,\ Q_2 = \frac{q-3}{4},\ \ Q_1 = \frac{q^3-51q+94}{144}).$

Therefore the rank is given by;

$$r = 2 + \frac{2q-4}{3} + \frac{q-3}{4} + \frac{q^3 - 51q + 94}{144}$$
$$= \frac{1}{144}[q^3 + 81q + 82].$$

Hence the subdegrees of G on the cosets of H are as shown in Table 5.2.4.6.

Table 5.2.4.6: Subdegrees of G on the cosets of $\boldsymbol{H = A_4}$ when $\boldsymbol{p} > 3$ $\boldsymbol{q \equiv -1\, mod\, 3}$

Suborbit Length	1	4	6	12
No. of Suborbits	2	$\frac{2q-4}{3}$	$\frac{q-3}{4}$	$\frac{q^3-51q+94}{144}$

5.2.5 Subdegrees of $\boldsymbol{G}$ on the cosets of $\boldsymbol{H = A_5}$

A_5 has the following nine conjugacy classes of subgroups;

H_{1-} Identity subgroup

H_{2-} 15 conjugate subgroups of order 2 isomorphic to C_2

H_{3-} 10 conjugate subgroups of order 3 isomorphic to C_3

H_{4-} 5 conjugate subgroups of order 4 isomorphic to V_4

H_{5-} 6 conjugate subgroups isomorphic to C_5

H_{6-} 10 conjugate subgroups isomorphic to D_6

H_{7-} 6 conjugate subgroups isomorphic to D_{10}

H_{8-} 5 conjugate subgroups isomorphic to A_4

H_{9-} A_5

(Kamuti, 1992, p. 93)

The corresponding table of marks of H omitting zeros above the main diagonal is as shown in Table 5.2.5.1 below.

Table 5.2.5.1: Table of Marks of $\boldsymbol{H = A_5}$

	$H_1 = I$	H_2	H_3	H_4	H_5	H_6	H_7	H_8	$H_9 = A_5$		
$H(/H_1)$	60										
$H(/H_2)$	30	2									
$H(/H_3)$	20	0	2								
$H(/H_4)$	15						0			3	3
$H(/H_5)$	12	0	0	0	2						
$H(/H_6)$	10	2	1	0	0	1					
$H(/H_7)$	6	2	0	0	1	0	1				
$H(/H_8)$	5	1	2	1	0	0	0	1			
$H(/H_9)$	1	1	1	1	1	1	1	1	1		

After computing the table of marks of H we now need to find $m(F) = m(F, H, G)$, where F is a representative of a conjugacy class in H and $F \leq H$.

Our computation will be carried out under the following 12 cases;

a) $p = 2$ $\quad q \equiv 4 \; mod \; 15$

b) $p = 2$ $\quad q \equiv 11 \; mod \; 15$

c) $p = 2$ $\quad q \equiv -1 \; mod \; 15$

d) $p = 2$ $q \equiv 1 \, mod \, 15$

e) $p = 3$ $q \equiv 1 \, mod \, 5$

f) $p = 3$ $q \equiv -1 \, mod \, 5$

g) $p = 5$ $q \equiv 1 \, mod \, 3$

h) $p = 5$ $q \equiv -1 \, mod \, 3$

i) $q \equiv 1 \, mod \, 30$

j) $q \equiv 11 \, mod \, 30$

k) $q \equiv -1 \, mod \, 30$

l) $q \equiv 19 \, mod \, 30$

Case a)

Also in this subgroup, to find $m(F)$ we first need to determine $|N_G(F)|$ and $|N_H(F)|$. We obtain $|N_G(F)|$ using Lemma 5.2.1.1, Lemma 5.2.1.2 and Lemma 5.2.1.3. We compute $|N_H(F)|$ using Lemma 1.2.3.2. Table 5.2.5.2 below gives the values of $m(F)$.

Table 5.2.5.2: The mark of F where $\boldsymbol{F \leq H = A_5}$ when $\boldsymbol{p = 2}$, $\boldsymbol{q \equiv 4 \, mod \, 15}$

F	$\lvert N_G(F)\rvert$	$\lvert N_H(F)\rvert$.	$m(F)$
$H_1 = \mathrm{I}$	$q(q^2-1)$	60	$\frac{q(q^2-1)}{60}$
$H_2 = \mathrm{C}_2$	q	4	$\frac{q}{4}$
$H_3 = \mathrm{C}_3$	$2(q-1)$	6	$\frac{(q-1)}{3}$
$H_4 = \mathrm{V}_4$	24	12	2
$H_5 = \mathrm{C}_5$	$2(q+1)$	10	$\frac{q+1}{5}$
$H_6 = \mathrm{D}_6$	6	6	1
$H_7 = \mathrm{D}_{10}$	10	10	1
$H_8 = \mathrm{A}_4$	24	12	2
$H_9 = \mathrm{A}_5$	60	60	1

Let $Q = (Q_{1,}Q_{2,}Q_{3,}Q_4, Q_5, Q_{6,}Q_{7,}Q_{8,}Q_9)$ denote the number of suborbits $\Delta_{\boldsymbol{j}}$. Then by Theorem 1.1.19 and using Table 5.2.5.1 and Table 5.2.5.2 we obtain the following system of equations;

$$60Q_1 + 30Q_2 + 20Q_3 + 15Q_4 + 12Q_5 + 10Q_6 + 6Q_7 + 5Q_8 + Q_9 = \frac{q(q^2-1)}{60}$$

$$2Q_2 + 3Q_4 + 2Q_6 + 2Q_7 + Q_8 + Q_9 = \frac{q}{4}$$

$$2Q_3 + Q_6 + 2Q_8 + Q_9 = \frac{q-1}{3}$$

$$3Q_4 + Q_8 + Q_9 = 2$$

$$2Q_5 + Q_7 + Q_9 = \frac{q+1}{5}$$

$$Q_6 + \quad Q_9 = 1$$

$$Q_7 \quad + Q_9 = 1$$

$$Q_8 + Q_9 = 2$$

$$Q_9 = 1.$$

Solving the above system of equations we obtain;

$$Q = \left(Q_9 = 1, Q_8 = 1,\ Q_7 = 0,\ \ Q_6 = 0, Q_5 = \frac{q-4}{10},\ \ Q_4 = 0, Q_3 = \frac{q-10}{6},\ \ \ Q_2 = \frac{q-8}{8},\ \ Q_1 = \frac{q^3 - 498q + 3728}{3600}\right).$$

Therefore the rank is given by;

$$r = 1 + 1 + \frac{q-4}{10} + \frac{q-10}{6} + \frac{q-8}{8} + \frac{q^3-498q+3728}{3600}$$

$$= \frac{q^3 + 912q - 112}{3600}.$$

Hence the subdegrees of *G* on the cosets of *H* are as shown in Table 5.2.5.3

Table 5.2.5.3: Subdegrees of *G* on the cosets of $\boldsymbol{H = A_5}$ when $\boldsymbol{p = 2}$ $\boldsymbol{q \equiv 4 \; mod \; 15}$

Suborbit Length	1	5	12	20	30	60
No. of Suborbits	1	1	$\frac{q-4}{10}$	$\frac{q-10}{6}$	$\frac{q-8}{8}$	$\frac{q^3 - 498q + 3728}{3600}$

NB: The procedure of finding the subdegrees for the remaining cases is the same as case a) so, we are going to quote the results only.

Case b)

$$Q = (Q_9 = 1, \quad Q_8 = 1, \quad Q_7 = 0, \quad Q_6 = 0, \; Q_5 = \frac{q-6}{10}, \; Q_4 = 0,$$

$$Q_3 = \frac{q-8}{6}, Q_2 = \frac{q-8}{8}, Q_1 = \frac{q^3 - 498q + 3472}{3600})$$

So the rank is given by;

$$r = 1 + 1 + \frac{q-6}{10} + \frac{q-8}{6} + \frac{q-8}{8} + \frac{q^3 - 498q + 3472}{3600}$$

$$= \frac{q^3 + 912q + 112}{3600}.$$

Hence the subdegrees of *G* on the cosets of *H* are as shown in Table 5.2.5.4

Table 5.2.5.4: Subdegrees of *G* on the cosets of $\boldsymbol{H = A_5}$ when $\boldsymbol{p = 2}$ $\boldsymbol{q \equiv -1 \; mod \; 15}$

Suborbit Length	1	5	12	20	30	60
No. of Suborbits	1	1	$\frac{q-6}{10}$	$\frac{q-8}{6}$	$\frac{q-8}{8}$	$\frac{q^3 - 498q + 3472}{3600}$

Case d)

$$Q = (Q_9 = 1, Q_8 = 1, Q_7 = 0, Q_6 = 0, Q_5 = \frac{q-6}{10}, \; Q_4 = 0, Q_3$$

$$= \frac{q-10}{6}, \quad Q_2 = \frac{q-8}{8}, Q_1 = \frac{q^3 - 498q + 3872}{3600}).$$

So the rank of G is given by;

$$r = 1 + 1 + \frac{q-6}{10} + \frac{q-10}{6} + \frac{q-8}{8} + \frac{q^3 - 498q + 3872}{3600}$$

$$= \frac{1}{3600}[q^3 + 912q - 688].$$

Hence the subdegrees of G on the cosets of H are as shown in Table 5.2.5.5

Table 5.2.5.5: Subdegrees of G on the cosets of $\boldsymbol{H = A_5}$ when $\boldsymbol{p = 2 \;\; q \equiv 1 \; mod \; 15}$

Suborbit Length	1	5	12	20	30	60
No. of Suborbits	1	1	$\frac{q-6}{10}$	$\frac{q-10}{6}$	$\frac{q-8}{8}$	$\frac{q^3 - 498q + 3862}{3600}$

Case e)

$$Q = \left(Q_9 = 1, \quad Q_8 = 1, \quad Q_7 = 0, \quad Q_6 = 0, \quad Q_5 = \frac{q-11}{20}, \; Q_4 = 0, \; Q_3 = \frac{q-18}{12}, \; Q_2 = \frac{q-9}{8}, Q_1 = \frac{q^3 - 362q + 3861}{3600}\right).$$

So the rank is as given below;

$$r = 1 + 1 + \frac{q-11}{20} + \frac{q-18}{12} + \frac{q-9}{8} + \frac{q^3 - 362q + 3861}{3600}$$

$$= \frac{1}{3600}[q^3 + 1137q - 738].$$

Hence the subdegrees of G on the cosets of H are as shown in Table 5.2.5.6

Table 5.2.5.6: Subdegrees of G on the cosets of $\boldsymbol{H = A_5}$ when $\boldsymbol{p = 3}$ $\boldsymbol{q \equiv 1\, mod 15}$

Suborbit Length	1	5	12	20	30	60
No. of Suborbits	1	1	$\frac{q-11}{20}$	$\frac{q-18}{12}$	$\frac{q-9}{8}$	$\frac{q^3-362q+3861}{3600}$

Case f)

$$Q = (Q_9 = 1, Q_8 = 1, Q_7 = 0, \quad Q_6 = 0, \quad Q_5 = \frac{q-4}{10}, \; Q_4 = 0, Q_3 = \frac{q-9}{6}, Q_2 = \frac{q-3}{4}, Q_1 = \frac{q^3-723q+3078}{3600}).$$

So the rank(r) of G is given by;

$$r = 1 + 1 + \frac{q-4}{10} + \frac{q-9}{6} + \frac{q-3}{4} + \frac{q^3-723q+3078}{3600}$$

$$= \frac{q^3+1137q+738}{3600}.$$

Hence the subdegrees of G on the cosets of H are as shown in Table 5.2.5.7

Table 5.2.5.7: Subdegrees of G on the cosets of $\boldsymbol{H = A_5}$ when $\boldsymbol{p = 3}$ $\boldsymbol{q \equiv -1\, mod\, 5}$

Suborbit Length	1	5	12	20	30	60
No. of Suborbits	1	1	$\frac{q-4}{10}$	$\frac{q-9}{6}$	$\frac{q-3}{4}$	$\frac{q^3-723q+3078}{3600}$

Case g)

$$Q = \left(Q_9 = 1, \quad Q_8 = 1, \quad Q_7 = 0, \quad Q_6 = 0, \quad Q_5 = \frac{q-5}{10}, \; Q_4 = 0, Q_3 = \frac{q-10}{6}, Q_2 = \frac{q-5}{4}, Q_1 = \frac{q^3 - 722q + 4250}{3600}\right).$$

So the rank(r) is given by

$$r = 1 + 1 + \frac{q-5}{10} + \frac{q-10}{6} + \frac{q-5}{4} + \frac{q^3 - 722q + 4250}{3600} = \frac{q^3 + 1137q - 850}{3600}.$$

Hence the subdegrees of G on the cosets of H are as shown in Table 5.2.5.8

Table 5.2.5.8:Subdegrees of *G* on the cosets of $\boldsymbol{H = A_5}$when $\boldsymbol{p = 5\ q \equiv 1 mod 3}$

Suborbit Length	1	5	12	20	30	60
No. of Suborbits	1	1	$\frac{q-5}{10}$	$\frac{q-10}{6}$	$\frac{q-5}{4}$	$\frac{q^3 - 722q + 4250}{3600}$

Case h)

$$Q = (Q_9 = 1, \quad Q_8 = 1, \quad Q_7 = 0, \; Q_6 = 0, \quad Q_5 = \frac{q-5}{10}, \; Q_4 = 0, Q_3 = \frac{q-8}{6}, Q_2 = \frac{q-3}{4}, Q_1 = \frac{q^3 - 723q + 2950}{3600}).$$

So the rank (r) of *G* is given by

$$r = 1 + 1 + \frac{q-5}{10} + \frac{q-8}{6} + \frac{q-3}{4} + \frac{q^3 - 723q + 2950}{3600}$$

$$= \frac{q^3 + 1137q + 850}{3600}.$$

Hence the subdegrees of G on the cosets of H are as shown in Table 5.2.5.9

Table 5.2.5.9:Subdegrees of *G* on the cosets of $\boldsymbol{H = A_5}$when $\boldsymbol{p = 5\ q \equiv 1 mod 3}$

Suborbit Length	1	5	12	20	30	60
No. of Suborbits	1	1	$\frac{q-5}{10}$	$\frac{q-8}{6}$	$\frac{q-3}{4}$	$\frac{q^3 - 723q + 2950}{3600}$

Case i)

$$Q = (Q_9 = 1, \quad Q_8 = 1, \quad Q_7 = 0, \; Q_6 = 0, \; Q_5 = \frac{q-6}{10}, \; Q_4 = 0, Q_3 = \frac{q-10}{3}, Q_2 = \frac{q-5}{4}, Q_1 = \frac{q^3 - 723q + 4322}{3600})$$

Therefore the rank (r) of *G* is as given as;

$$r = 1 + 1 + \frac{q-6}{10} + \frac{q-10}{3} + \frac{q-5}{4} + \frac{q^3 - 723q + 4322}{3600}$$

$$= \frac{q^3 + 1137q - 1138}{3600}.$$

Hence the subdegrees of G on the cosets of H are as shown in Table 5.2.5.10

Table 5.2.5.10:Subdegrees of G on the cosets of $\boldsymbol{H = A_5}$when $\boldsymbol{p = 2}$ $\boldsymbol{q \equiv 1\ mod\ 15}$

Suborbit Length	1	5	12	20	30	60
No. of Suborbits	1	1	$\frac{q-6}{10}$	$\frac{q-10}{3}$	$\frac{q-5}{4}$	$\frac{q^3-723q+4322}{3600}$

Case j)

$$Q = \left(Q_9 = 1, \quad Q_8 = 1, \quad Q_7 = 0, \quad Q_6 = 0, \quad Q_5 = \frac{q-6}{10}, \ Q_4 = 0, Q_3 = \frac{q-8}{6}, Q_2 = \frac{q-5}{4}, Q_1 = \frac{q^3-723q+3922}{3600}\right).$$

So the rank (r) of G is given below

$$r = 1 + 1 + \frac{q-6}{10} + \frac{q-8}{6} + \frac{q-5}{4} + \frac{q^3-723q+3922}{3600}$$

$$= \frac{q^3+1137q-338}{3600}.$$

Hence the subdegrees of G on the cosets of H are as shown in table 5.2.5.11

Table 5.2.5.11: Subdegrees of G on the cosets of $\boldsymbol{H = A_5}$ where $\boldsymbol{q \equiv 11\ mod\ 15}$

Suborbit Length	1	5	12	20	30	60
No. of Suborbits	1	1	$\frac{q-6}{10}$	$\frac{q-8}{6}$	$\frac{q-5}{4}$	$\frac{q^3-723q+3922}{3600}$

Case k)

$$Q = \left(Q_9 = 1, \quad Q_8 = 1, \quad Q_7 = 0, \quad Q_6 = 0, Q_5 = \frac{q-4}{10}, \ Q_4 = 0, Q_3 = \frac{q-8}{6}, Q_2 = \frac{q-3}{4}, Q_1 = \frac{q^3 - 723q + 2878}{3600}\right).$$

The rank (r) of G is given by;

$$r = 1 + 1 + \frac{q-4}{10} + \frac{q-8}{6} + \frac{q-3}{4} + \frac{q^3 - 723q + 2878}{3600}$$
$$= \frac{q^3 + 1137q + 1138}{3600}.$$

Hence the subdegrees of G on the cosets of H are as shown in Table 5.2.5.12

Table 5.2.5.12: Subdegrees of G on the cosets of $\boldsymbol{H = A_5}$ where $\boldsymbol{q \equiv -1 \ mod\ 15}$

Suborbit Length	1	5	12	20	30	60
No. of Suborbits	1	1	$\frac{q-4}{10}$	$\frac{q-8}{6}$	$\frac{q-3}{4}$	$\frac{q^3-723q+2878}{3600}$

Case l)

$$Q = (Q_9 = 1, \quad Q_8 = 1, \quad Q_7 = 0, \quad Q_6 = 0, \quad Q_5 = \frac{q-4}{10}, Q_4 = 0, Q_3 = \frac{q-10}{6}, Q_2 = \frac{q-5}{4}, Q_1 = \frac{q^3 - 723q + 4178}{3600}).$$

Therefore the rank (r) G is as shown below;

$$r = 1 + 1 + \frac{q-4}{10} + \frac{q-10}{6} + \frac{q-5}{4} + \frac{q^3-723q+4178}{3600}$$
$$= \frac{q^3 + 1137q - 562}{3600}.$$

Hence the subdegrees of G on the cosets of H are as shown in Table 5.2.5.13

Table 5.2.5.13: Subdegrees of *G* on the cosets of $\boldsymbol{H = A_5}$ where $\boldsymbol{q \equiv 19\ mod\ 30}$

Suborbit Length	1	5	12	20	30	60
No. of Suborbits	1	1	$\frac{q-4}{10}$	$\frac{q-10}{6}$	$\frac{q-5}{4}$	$\frac{q^3-723q+4178}{3600}$

5.2.6 Subdegrees of $\boldsymbol{G}$ on the cosets of $\boldsymbol{H = D_{2(q-1)}}$

We shall compute the subdegrees of *G* in this representations under the following two cases:

a) When q is even

b) When q is odd

Case a)

When q is even

The conjugacy classes of subgroups are;

i) H_1 − Identity.

ii) H_2 − a conjugacy class of $q-1$ cyclic subgroups of order two

iii) H_3 −Normal cyclic subgroups H_{3_1}, $H_{3_2}, \ldots, H_{3_r}$ contained in C_{q-1} where $m_i \mid q-1$ and $m_i \neq 2$, $1 \leq i \leq r$.

iv) H_4 −Dihedral subgroups H_{4_1}, $H_{4_2}, \ldots, H_{4_r}$ where $m_i \mid q-1$, $1 \leq i \leq r$.

v) H_5 $-$A normal cyclic subgroup of order $q-1$, C_{q-1}.

vi) $H_6 - D_{2(q-1)}$.

The corresponding table of marks of $D_{2(q-1)}$,when q is even is as shown in Table 5.2.6.1

Table 5.2.6.1: Table of marks of $H =$ $D_{2(q-1)}$, $\boldsymbol{q}$ even

	H_1	H_2	H_{3_1} ..	H_{3_1}	H_{4_1} .	H_{4_r}	H_5	H_6
$H(/H_1)$	$2(q-1)$							
$H(/H_2)$	$q-1$	1						
$H(/H_{3_1})$	m_{31}	m_{32}	m_{33}					
$\vdots$	..	…	….	….				
$H(/H_{3_r})$	m_{r+21}	m_{r+22}	m_{r+23}	……	m_{r+2r+}			
$H(/H_{4_1})$	m_{r+31}	m_{r+32}	m_{r+33}	m_{r+34}	m_{r+35}			
$\vdots$	…..	….	….	….	….	….		
$H(/H_{4_r})$	m_{2r+21}	m_{2r+22}	m_{2r+23}	m_{2r+24}	….	…	….	
$H(/H_5)$	2	0	0	2	2	…	2	0
$H(/H_6)$	1	1	1	1	1	…	1	1

We now proceed to compute $m(F) = m(F, H, G)$. To find $m(F)$ we first need to determine $|N_G(F)|$ and $|N_H(F)|$. We obtain $|N_G(F)|$ using Lemma 5.2.1.1, Lemma 5.2.1.2 and Lemma 5.2.1.3. The values of $m(F)$ are displayed in Table 5.2.6.2 below;

Table 5.2.6.2: The mark of F where $\boldsymbol{F \leq H =}\ D_{2(q-1)}$, $\boldsymbol{q}$ **even**

F	$\|N_G(F)\|$	$\|N_H(F)\|$.	$m(F)$
H_1	$q(q^2-1)$	$2(q-1)$	$\frac{q(q+1)}{2}$
H_2	q	2	$\frac{q}{2}$
H_{3_1}	$2(q-1)$	$2(q-1)$	1
$\vdots$	$\cdots$	$\cdots$	$\cdots$
H_{3_r}	$2(q-1)$	$2(q-1)$	1
H_{4_1}	*2m*	*2m*	1
$\vdots$	...	...	$\cdots$
H_{4_r}	*2m*	*2m*	1
H_5	$2(q-1)$	$2(q-1)$	1
H_6	$2(q-1)$	$2(q-1)$	1

Let $Q=\left(Q_1,Q_2,Q_3,\ldots,Q_{r+2},Q_{r+3},\ldots,Q_{2r+2},Q_{2r+3},Q_{2r+4}\right)$ denote the number of suborbits Δ_j. Then by Theorem 1.1.19 and using Table 5.2.6.1 and Table 5.2.6.2 we obtain the following system of equations;

$$
\begin{aligned}
2(q-1)Q_1+(q-1)Q_2+Q_3m_{31}+\ldots+Q_{r+2}m_{r+21} \quad + \quad +\ldots \quad +Q\alpha_{2r+3} \quad +Q_{2r+4} &= \frac{q(q+1)}{2} \\
Q_2+Q_3m_{32}+\ldots+Q_{r+2}m_{r+22} \quad +\ldots+Q_{2r+2}m_{2r+22} \qquad +Q_{2r+4} &= \frac{q}{2} \\
Q_3m_{33}+\ldots+Q_{r+2}m_{r+23} \quad +Q_{r+3}\ m_{r+33} \quad +\ldots \quad +2_{2r+3} \quad +Q_{2r+4} &= 1 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
Q_{r+2}m_{r+2r+2} \quad +Q_{r+3}m_{r+3+r+2}+\ldots+2\alpha_{2r+3} \quad +Q_{2r+4} &= 1 \\
Q_{r+3}m_{r+3+r+3}+\ldots \qquad +Q_{2r+4} &= 1 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
Q_{2r+2}m_{2r+22r+2} \qquad +Q_{2r+4} &= 1 \\
Q\alpha_{2r+3} \quad +Q_{2r+4} &= 1 \\
Q_{2r+4} &= 1.
\end{aligned}
$$

Solving the above systems of equations we obtain;

$$Q=(Q_1=1,Q_2=\frac{q-2}{2},Q_3=0,.,Q_{r+2}=0,Q_{r+3}=0,.,Q_{2r+2}=0,Q_{2r+3}=0,Q_{2r+4}=1).$$

So the rank (r)

$$
\begin{aligned}
r &= 1+\frac{q-2}{2}+1 \\
&= \frac{q+2}{2}
\end{aligned}
$$

Hence the subdegrees of G on the cosets of H are as shown below;

Table 5.2.6.3: Subdegrees of G on the cosets of $H = D_{2(q-1)}$, $\boldsymbol{q}$ even

Suborbit Length	1	$q-1$	$2(q-1)$
No. of Suborbits	1	$\frac{q-2}{2}$	1

When q is odd

$$Q = (Q_1 = 1, Q_2 = \frac{q-3}{2}, Q_3 = 1, , Q_{r+2} = 0, Q_{r+3} = 0, ., Q_{2r+2} = 0, Q_{2r+3} = 0, Q_{2r+4} = 1).$$

So the rank (r) is as shown below

$$r = 1 + 1 + \frac{q-3}{2} + 1$$

$$= \frac{q+3}{2}.$$

Hence the subdegrees of G on the cosets of H are as shown in Table 5.2.6.4 below;

Table 5.2.6.4: Subdegrees of G on the cosets of $H = D_{2(q-1)}$, $\boldsymbol{q}$ odd

Suborbit Length	1	$\frac{q-1}{2}$	$q-1$	$2(q-1)$
No. of Suborbits	1	1	$\frac{q-3}{2}$	1

CHAPTER SIX

SUBORBITAL GRAPHS CORRESPONDING TO PERMUTATION REPRESENTATIONS OF $PGL(2, q)$

After having computed the ranks and the subdegrees of a permutation group acting on the cosets of its subgroup; the next most interesting thing is to construct and investigate some properties of suborbital graphs corresponding to these actions.

This chapter has three sections. Section 6.1 gives the background information which will be used later in the chapter. In Section 6.2 we determine suborbits of G acting on the cosets of C_{q-1}. Finally in Section 6.3 we construct suborbital graphs Γ_i of G corresponding to the suborbits of length $q-1$ when G acts on the cosets of C_{q-1}.

6.1 Suborbital Graphs

6.1.1 Definition

Let G be a transitive permutation group acting on a set X. Then G acts on $X \times X$ by $g(x,y)=(gx,gy)$, $g \in G, x, y \in X$. If $O \subseteq X \times X$ is a G-Orbit on $X \times X$, then for a fixed $x \in X$, $\Delta = \{y \in X \,|\, (x.y) \in O\}$ is a G_x-Orbit. Conversely, if $\Delta \subset X$ is a G_x-Orbit, then $O = \{(gx, gy) \,|\, g \in G, \, y \in \Delta\}$ is a G-Orbit on $X \times X$, We say Δ corresponds to O. The G-orbits on $X \times X$ are called suborbitals. Let

$O_i \subseteq X \times X$, $i = 0,1,...,r-1$ be a suborbital. Then we form a graph Γ_i, by taking X as the set of vertices of Γ_i and by including a directed edge from x to y $(x, y \in X)$ if and only if $(x, y) \in O_i$. Thus each suborbital O_i determines a suborbital graph Γ_i. Now $O_i^* = \{(x, y) | (y, x) \in O_i\}$ is also a *G*-Orbit. Let Γ_i^* be the suborbital graph corresponding to the suborbital O_i^*. Let the suborbits Δ_i $(i = 0,1,...,r-1)$ correspond to the suborbital O_i. Then Γ_i is undirected if Δ_i is self-paired and directed if Δ_i is not self-paired. (Sims, 1967)

6.1.2 Lemma

Let *G* be a transitive permutation group acting on *X*. Then there are bijections between;

a) The set of orbits of G_x on *X* for a fixed $x \in X$.

b) The set of orbits of *G* on $X \times X$.

c) The set of double cosets $G_x g G_x$, $g \in G$ for a fixed $x \in X$.

(Kamuti, 1992)

6.1.3 Theorem

Let *G* be transitive on X. Then *G* is primitive if and only if suborbital graph Γ_i $i = 1,2, ..., r-1$ is connected. (Sims, 1967)

6.1.4 Theorem

Let G act transitively on a set X. Let $x \in X$ and $H = G_x$. Then the action of G on X is equivalent to the action by multiplication on the set of (right) cosets of H in G. (Rose,1978, p. 76)

6.1.5 Theorem

Let G be transitive on X and let G_x be the stabilizer of a point $x \in X$. Let

$\Delta_0 = \{x\}, \Delta_{1,}\ \Delta_{2,}$………………,$\Delta_{k-1}$ be orbits of G_x on X of length;

$n_0 = 1, n_1, n_2, ., ., n_{k-1}$, where $n_0 \leq\ n_1 \leq\ n_2 \leq, ., ., \leq n_{k-1}$. If there exists an index $j > 0$ such that $n_j > n_1 n_{j-1}$, then G is primitive. (Wielandt, 1964)

6.1.6 Lemma

Let G act on a set X. Then the character π of a permutation representation of $g \in G$ on X is defined by;

$$\pi(g) = |Fix(g)|, \forall\ g \in G$$

6.1.7 Theorem

Let G act transitively on a set X, and let $g \in G$. Suppose π is the character of the permutation representation of G on X, then the number of self-paired suborbits of G is given by;

$$n_\pi = \frac{1}{|G|} \sum_{g \in G} \pi(g^2).$$

(Cameron, 1975)

6.2 Suborbits of $G = PGL(2, q)$ acting on the cosets of $H = C_{q-1}$

Let G act on the projective line $PG(1, q) = GF(q) \cup \{\infty\}$. If $\tau_{0,}\ \tau_1,\ \tau_2$ is the set of elliptic, parabolic and hyperbolic elements respectively, the cycle structures of $g \in G$ is as given in the table below.

Table 6.2.1: The cycle structures of $\boldsymbol{g} \in \boldsymbol{G}$

	τ_1	τ_2	τ_0
Cycle length of g	1 p	1 d	d
No. of cycles	1 p^{f-1}	2 $\frac{q-1}{d}$	$\frac{q+1}{d}$

where d is the order of $g \in G$.

Since $G$is 2-transitive on the $PG(1, q)$, then it is transitive on ordered pairs from $PG(1, q)$.

Because H is the stabilizer of an ordered pair, we deduce the following from Theorem 6.1.4;

6.2.1 Corollary

The action of G on the cosets of H is equivalent to its action on the ordered 2- element subsets from the projective line.

Proof

Using Theorem 6.1.4 the result is immediate. ■

From now henceforth we shall be working on the action of G on the ordered pairs from the projective line where $PG(1,q) = \{\infty, 0, 1, \beta, \beta^2, \dots \dots \dots \dots, \beta^{q-2}\}$. Let H be the stabilizer of $(\infty, 0)$ and β be a primitive element of $GF(q)$, then

$$H \equiv < \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} > \equiv (0)(\infty)(1\ \beta\ \beta^2\ \dots \dots \dots \dots\ \beta^{q-2}).$$

So the suborbits of G on ordered pairs from $PG(1,q)$ are;

$$Orb_{G_{(\infty,0)}}(\infty, 0) = \{(\infty, 0)\} = \Delta_0$$

$$Orb_{G_{(\infty,0)}}(0, \infty) = \{(0, \infty)\} = \Delta_1$$

$$Orb_{G_{(\infty,0)}}(\infty, 1) = \{(\infty, 1), (\infty, \beta), (\infty, \beta^2), \dots \dots, (\infty, \beta^{q-2})\} = \Delta_2$$

$$Orb_{G_{(\infty,0)}}(1, \infty) = \{(1, \infty), (\beta, \infty), (\beta^2, \infty), \dots \dots, (\beta^{q-2}, \infty)\} = \Delta_3$$

$$Orb_{G_{(\infty,0)}}(0,1) = \{(0,1), (0, \beta), (0, \beta^2), \dots \dots, (0, \beta^{q-2})\} = \Delta_4$$

$$Orb_{G_{(\infty,0)}}(1,0) = \{(1, 0), (\beta, 0), (\beta^2, 0), \dots \dots, (\beta^{q-2}, 0)\} = \Delta_5$$

$$Orb_{G_{(\infty,0)}}(1, \beta) = \{(1, \beta), (\beta, \beta^2), (\beta^2, \beta^3), \dots \dots, (\beta^{q-2}, 1)\} = \Delta_6$$

$$Orb_{G_{(\infty,0)}}(1, \beta^2) = \{(1, \beta^2), (\beta, \beta^3), (\beta^2, \beta^4), \dots \dots, (\beta^{q-2}, \beta)\} = \Delta_7$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$Orb_{G_{(\infty,0)}}(1, \beta^{q-2}) = \{(1, \beta^{q-2}), (\beta, 1), (\beta^2, \beta), \dots \dots, (\beta^{q-2}, \beta^{q-3})\} = \Delta_{q+3}$$

So the subdegrees are as shown in Table 6.2.2 below

Table 6.2.2: subdegrees of G on the ordered 2- element subsets from the $\boldsymbol{PG(1,q)}$

Suborbit length	1	$q-1$
No. of suborbits	2	$q+2$

6.2.2 Theorem

The action of G on the set of ordered pairs from $PG(1,q)$ is imprimitive.

Proof

Using Theorem 6.1.5 the result is immediate.■

6.2.3 Theorem

When G acts on the set of ordered pairs from $PG(1,q)$, the number of self paired suborbits is $q+2$ and the paired suborbits are 2.

Proof

If $g \in G$, for g^2 to fix an ordered pair then either g is the identity or g is a hyperbolic element of order 2 or g is a hyperbolic element of order greater than two or g is an elliptic element of order 2. If g is the identity then g^2 fixes $q(q+1)$ elements. If g is a hyperbolic element of order 2 then g^2 fixes $q(q+1)$ elements. There are $\frac{q(q+1)}{2}$ hyperbolic elements of order 2 in G hence contributes

$q(q+1)(\frac{q(q+1)}{2})$ number of fixed points to the formula in Theorem 6.1.6. If g is a hyperbolic element of order greater than 2 then g^2 fixes 2 ordered pairs (namely $(\infty,0)$ and $(0,\infty)$). In total we have $(q-3)(\frac{q(q+1)}{2})$ elements hence contribute

$2(q-3)(\frac{q(q+1)}{2})$ to the formula. If g is an elliptic element of order 2 then g^2 fixes $q(q+1)$ elements. In total we have $(\frac{q(q-1)}{2})$ elliptic elements of order 2 hence contribute $q(q+1)(\frac{q(q-1)}{2})$ to the formula. Now applying Theorem 6.1.6 we have;

$$n_{\pi}=\frac{1}{|G|}\left[q(q+1)+0+2(q-3)\left(\frac{q(q+1)}{2}\right)+q(q+1)\left(\frac{q(q+1)}{2}\right)\right.$$
$$\left.+q(q+1)\left(\frac{q(q-1)}{2}\right)\right]=q+2.\ \blacksquare$$

Since the rank of *G* on the cosets of *H* is $q+4$, the number of paired suborbits are 2.

6.2.4 Theorem

When *G* acts on the set of ordered pairs from $PG(1,q)$, then Δ_3 is paired with Δ_4.

Proof

Let $g \in G$, where $g=\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$, then g maps $(1,\infty) \in \Delta_3$ to $(\infty,0)$ and $(\infty,0)$ to $(0,-1) \in \Delta_4$. Therefore by Definition 6.1.1 Δ_3 and Δ_4 are paired. $\blacksquare$

6.3 Suborbital graphs Γ of *G* corresponding to its suborbits when *G* acts on the cosets of $H = C_{q-1}$

We recall from Section 6.2 that there are $q+2$ suborbits of length $q-1$ and 2 suborbits of length 1.

6.3.1 Suborbital graph corresponding to suborbit of *G* formed by pairs of the form $(\boldsymbol{\beta^i}, \infty)$

Since *G* is doubly transitive on the $PG(1,q)$; given a pair (v,h) $v \neq h$ v, $h \in PG(1,q)$, there exist $g \in G$ such that $g(\infty) = v$ and $g(0) = h$. For $v, h \neq \infty$, g can be chosen to be $\begin{pmatrix} v & h \\ 1 & 1 \end{pmatrix}$. If either v or h is ∞ then we can choose g to be $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix}$ respectively.

Now let β be the generator of $GF(q)^*$, then we construct suborbital graph as follows;

6.3.1.1 Theorem

For each of the following cases $((v,h),(c,d))$ is an edge in $\Gamma_{(\beta^i,\infty)}$.

a) $v, h \neq \infty$, $d = v$ and $c = \left(v\beta^i + h\right)(\beta^i + 1)^{-1}$

b) $d = v = \infty$ and $c = \beta^i + h$

c) $h = \infty$, $d = v$ and $c = (v\beta^i + 1)\beta^{-i}$

Proof

a) Since $\left((\infty, 0), \left(\beta^i, \infty\right)\right)$ is in $\Gamma_{(\beta^i,\infty)}$ so if $\left((v,h),(c,d)\right)$ is in $\Gamma_{(\beta^i,\infty)}$, then there exists $g \in G$ which send ∞ to v and 0 to h. So we choose $g = \begin{pmatrix} v & h \\ 1 & 1 \end{pmatrix}$ since $v, h \neq \infty$. Now;

$$g\left(\beta^i, \infty\right) = (c,d).$$

Hence;

$$\begin{pmatrix} v & h \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \beta^i \\ 1 \end{pmatrix} = \left(v\beta^i + h\right)(\beta^i + 1)^{-1} = c$$

and

$$\begin{pmatrix} v & h \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} v \\ 1 \end{pmatrix} = v = d.$$

b) Since $v = \infty$, then we choose $g \in G$ to be $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$, hence we have,

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \beta^i \\ 1 \end{pmatrix} = \left(\beta^i + h\right) = c$$

and

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \infty = d.$$

Hence $c = \left(\beta^i + h\right)$ and $d = \infty$.

c) $h = \infty$, so we choose $g \in G$ to be $\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix}$, hence we have,

$$g\left(\beta^i, \infty\right) = (c, d)$$

$$\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \beta^i 1 \\ 1\ 0 \end{pmatrix} = \begin{pmatrix} v\beta^i + 1 & v \\ \beta^i & 1 \end{pmatrix}.$$

Thus $d = v$ and $c = (v\beta^i + 1)\beta^{-i}$. ■

6.3.1.2 Example

Suborbital graph of $G = PGL(2,3)$

To construct suborbital graph corresponding to this action we first need to find suborbits of G using Corollary 6.2.1.

Thus we have;

$$Orb_{G_{(\infty,0)}}(\infty,0) = \{(\infty,0)\} = \Delta_0.$$

$$Orb_{G_{(\infty,0)}}(0,\infty) = \{(0,\infty)\} = \Delta_1$$

$$Orb_{G_{(\infty,0)}}(\infty,1) = \{(\infty,1),(\infty,2)\} = \Delta_2$$

$$Orb_{G_{(\infty,0)}}(1,\infty) = \{(1,\infty),(2,\infty)\} = \Delta_3$$

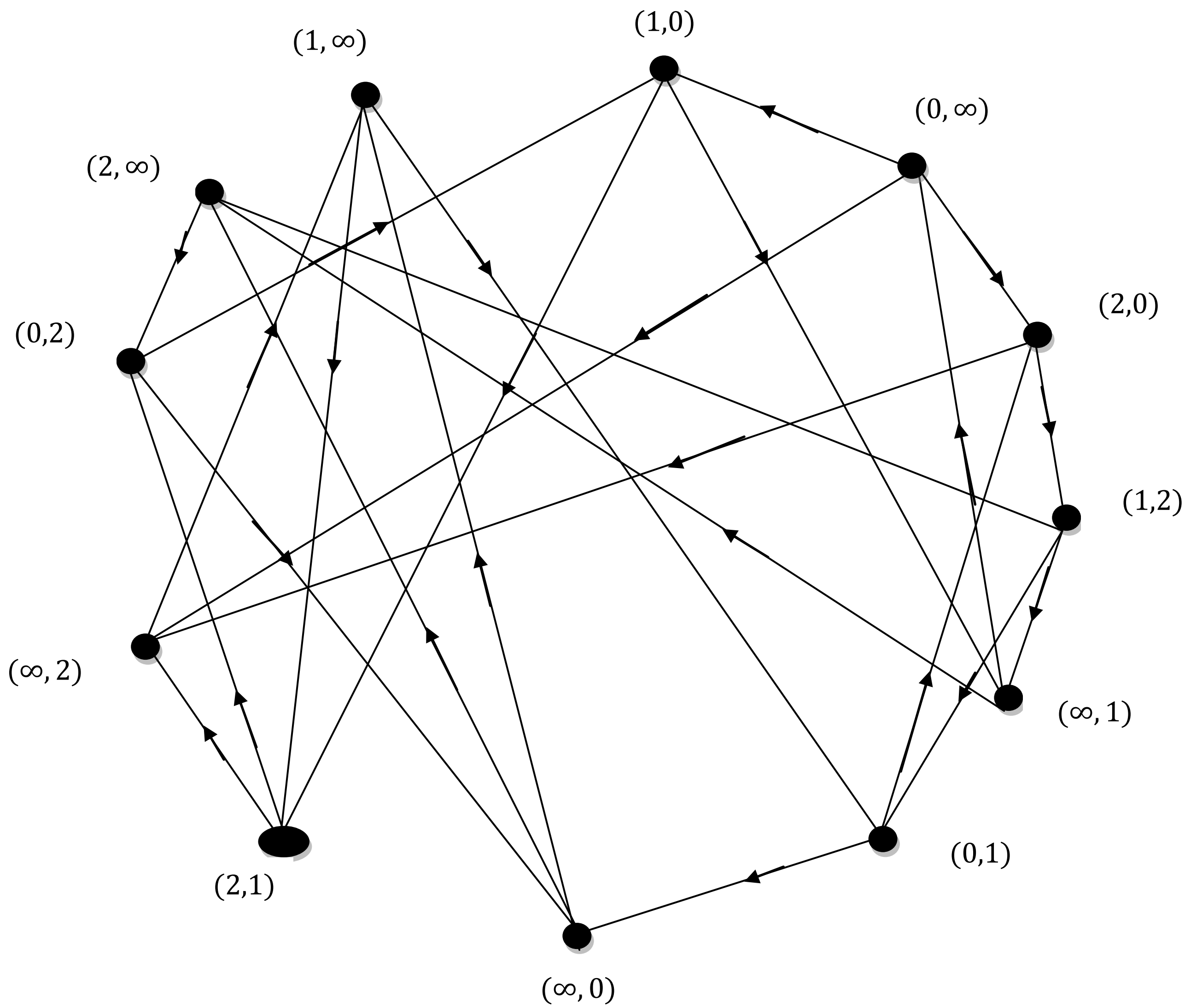
$$Orb_{G_{(\infty,0)}}(0,1) = \{(0,1),(0,2)\} = \Delta_4$$

$$Orb_{G_{(\infty,0)}}(1,0) = \{(1,0),(2,0)\} = \Delta_5$$

$$Orb_{G_{(\infty,0)}}(1,2) = \{(1,2),(2,1)\} = \Delta_6.$$

Then suborbits of *G* in this case are 7 and the subdegrees are 1,1,2,2,2,2,2. This is expected since from the results in Section 6.2 we have 2 orbits of length 1 and $q+2$ suborbits of length $q-1$. Using Theorem 6.3.1.1 we can now construct suborbital graph $\Gamma_{(\beta^i,\infty)}$ corresponding to Δ_3.

Figure 6.3.1.1: suborbital graph formed by pairs of the form $(\boldsymbol{\beta^i}, \infty)$



6.3.2 Suborbital graph corresponding to suborbit of G formed by pairs of the form $(\boldsymbol{\beta^i}, \mathbf{0})$

6.3.2.1 Theorem

For each of the following cases $((v,h),(c,d))$ is an edge in $\Gamma_{(\beta^i,0)}$.

a) $v,h \neq \infty$, $d = h$ and $c = (v\beta^i + h)(\beta^i + 1)^{-1}$

b) $d = h = \infty$ and $c = (v\beta^i + 1)\beta^{-i}$

c) $v = \infty$, $d = h$ and $c = \beta^i + h$

Proof

a) Using the same approach as Theorem 6.3.1.1, when $v,h \neq \infty$, we let $g \in G$ be $\begin{pmatrix} v & h \\ 1 & 1 \end{pmatrix}$. So

$$g(\beta^i, 0) = (c,d).$$

Hence;

$$\begin{pmatrix} v & h \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \beta^i \\ 1 \end{pmatrix} = (v\beta^i + h)(\beta^i + 1)^{-1} = c$$

and

$$\begin{pmatrix} v & h \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} h \\ 1 \end{pmatrix} = h = d.$$

Thus $c = (v\beta^i + h)(\beta^i + 1)^{-1}$ and $d = h$.

b) Since $h = \infty$, then we have,

$$\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \beta^i & 0 \\ 1 & 1 \end{pmatrix} = ((v\beta^i + 1)\beta^{-i}, \infty) = (c,d)$$

Hence $c = (v\beta^i + 1)\beta^{-i}$ and $d = \infty$.

c) $v = \infty$, then taking $g \in G$ to be $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$, hence we have,

$$g(\beta^i, 0) = (c, d)$$

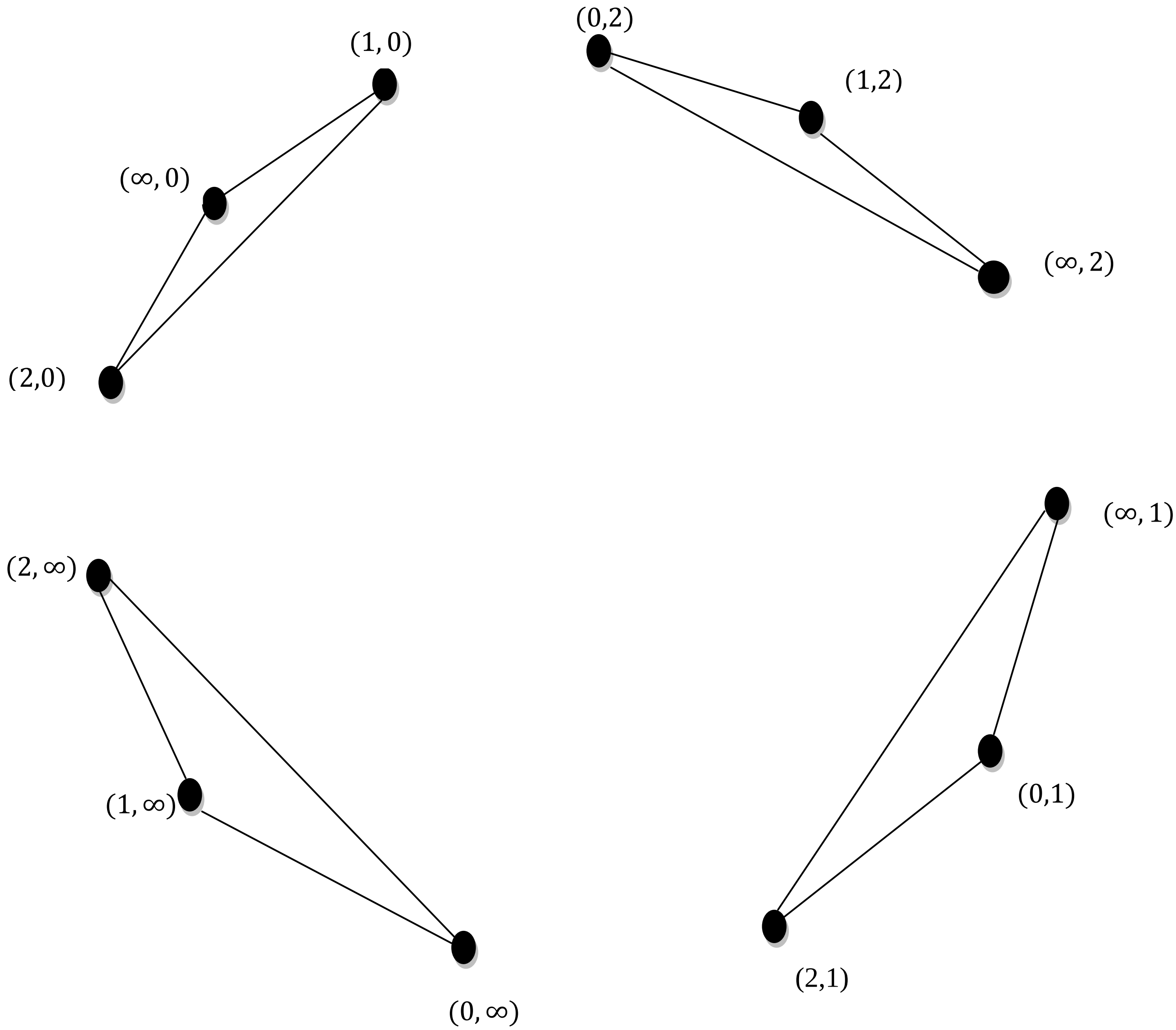
$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \beta^i & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \beta^i + h & h \\ 1 & 1 \end{pmatrix}$$

Thus $d = h$ and $c = \beta^i + h$■

6.3.2.2 Example

From Theorem 6.3.2.1 we can construct suborbital graph $\Gamma_{(\beta^i,0)}$ for $PGL(2,3)$ corresponding to suborbit Δ_5 (in Example 6.3.1.2**).**

Figure 6.3.2.1: Suborbital graph formed by pairs of the form $(\boldsymbol{\beta^i}, \mathbf{0})$



6.3.3 Suborbital graph corresponding to suborbit of G formed by pairs of the form $(\mathbf{0},\ \boldsymbol{\beta^i})$

6.3.3.1 Theorem

For each of the following cases $((v,h),(c,d))$ is an edge in $\Gamma_{(0,\beta)}$.

a) $v,h \neq \infty$, $c = h$ and $d = (v\beta^i + h)(\beta^i + 1)^{-1}$.

b) $v = \infty$, $c = h$ and $d = \beta^i + h$.

c) $h = c = \infty$, and $d = (v\beta^i + 1)\beta^{-i}$.

Proof

a) Since $v,h \neq \infty$, then we take $g \in G$ to be $\begin{pmatrix} v & h \\ 1 & 1 \end{pmatrix}$. So we have;

$$g(0,\beta^i) = (c,d).$$

Hence;

$$\begin{pmatrix} v & h \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & \beta^i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} h & v\beta^i + h \\ 1 & \beta^i + 1 \end{pmatrix} = (\,h,(v\beta^i + h)(\beta^i + 1)^{-1})$$

Thus $c = h$, $d = (v\beta^i + h)(\beta^i + 1)^{-1}$.

b) For $v = \infty$, so taking $g \in G$ to be $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$, so we have;

$$g(0,\beta^i) = (c,d)$$

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \beta^i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} h & \beta^i + h \\ 1 & 1 \end{pmatrix}.$$

So $c = h$ and $d = \beta^i + h$.

c) Finally for $h = \infty$, then we have,

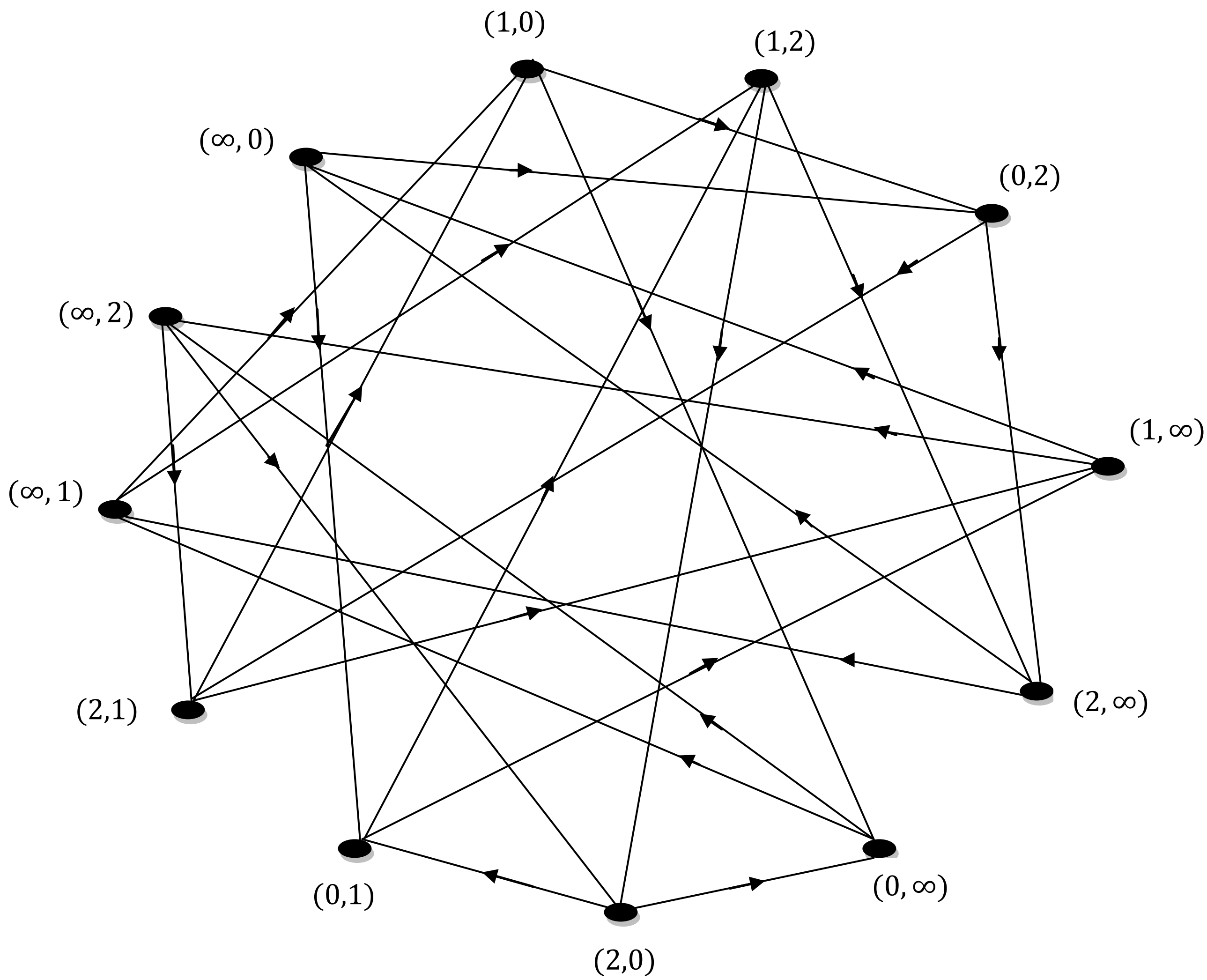
$$\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \beta^i \\ 1 & 1 \end{pmatrix} = \left(\infty, (v\beta^i + 1)\beta^{-i}\right) = (c, d).$$

Thus $c = \infty$ and $d = (v\beta^i + 1)\beta^{-i}$. ■

6.3.3.2 Example

Using Theorem 6.3.3.1 we can construct suborbital graph $\Gamma_{(0,\beta^i)}$ for $PGL(2,3)$ corresponding to suborbit Δ_4 (in Example 6.2.2.2).

Figure 6.3.3.1 Suborbital graph formed by pairs of the form $(\mathbf{0},\ \boldsymbol{\beta^i})$



6.3.4 Suborbital graph corresponding to suborbit of G formed by pairs of the form $(\infty, \boldsymbol{\beta^i})$

6.3.4.1 Theorem

For each of the following cases$((v,h),(c,d))$ is an edge in $\Gamma_{(\infty,\beta^i)}$.

a) $v,h \neq \infty$, $c = v$ and $d = (v\beta^i + h)(\beta^i + 1)^{-1}$

b) $v = c$ and $d = (v\beta^i + 1)\beta^{-i}$

c) $v = c = \infty$, and $d = \beta^i + h$.

Proof

a) $v,h \neq \infty$, so we take $g \in G$ to be $\begin{pmatrix} v & h \\ 1 & 1 \end{pmatrix}$; so

$$g(\infty, \beta^i) = (c,d)$$

Hence;

$$\begin{pmatrix} v & h \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & \beta^i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} v & v\beta^i + h \\ 1 & \beta^i + 1 \end{pmatrix} = (\, v, (v\beta^i + h)(\beta^i + 1)^{-1}).$$

Thus $c = v, \quad d = (v\beta^i + h)(\beta^i + 1)^{-1}$.

b) For $h = \infty$, In this case we take $g \in G$ to be $\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix}$, so we have;

$$g(\infty, \beta^i) = (c,d)$$

$$\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \beta^i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} v & \beta^i + 1 \\ 1 & \beta^i \end{pmatrix}$$

So $c = v$ and $d = (\beta^i + 1)\beta^{-i}$.

c) Finally for $v = \infty$, then we have,

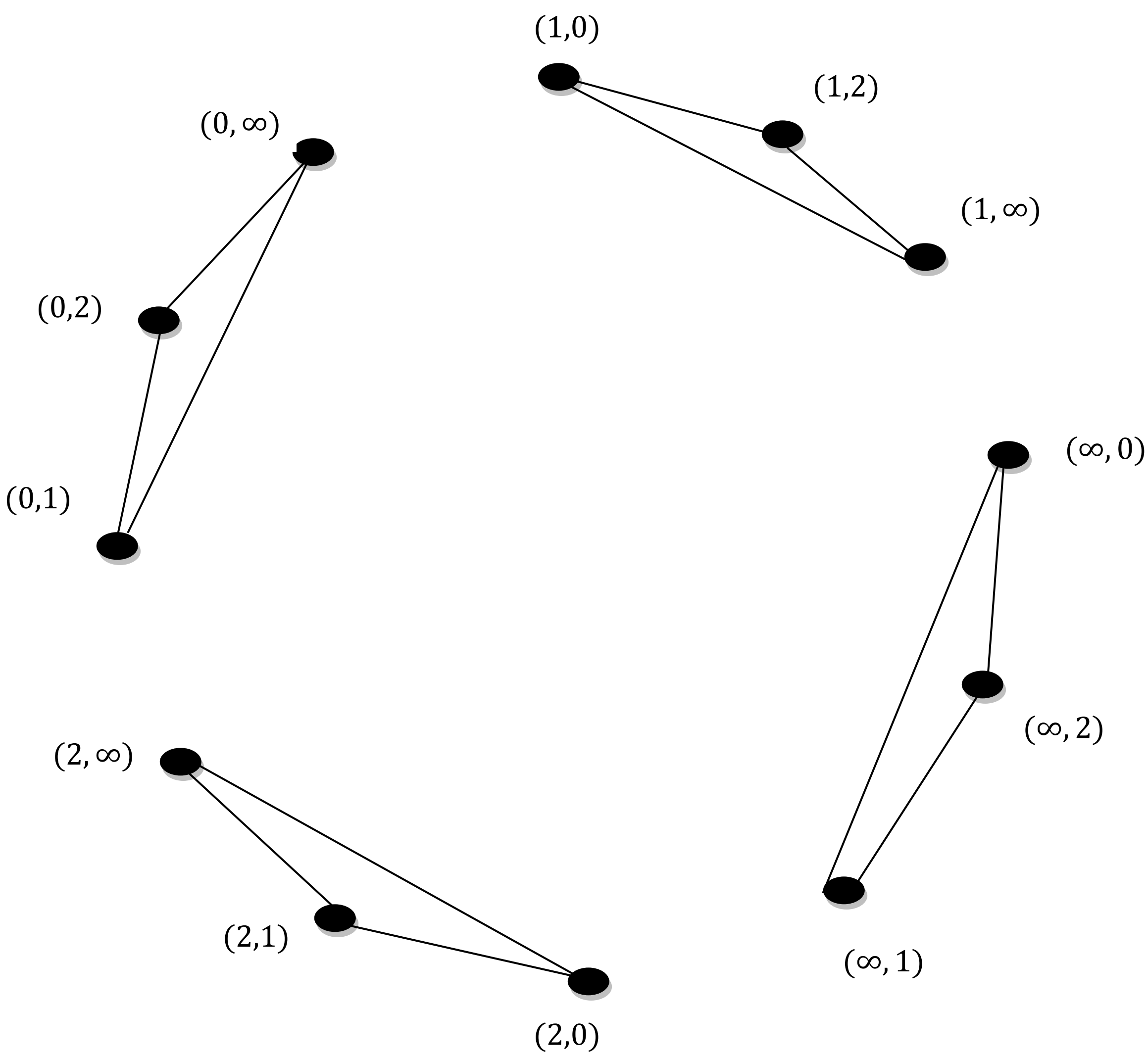
$$\begin{pmatrix}1 & h\\0 & 1\end{pmatrix}\begin{pmatrix}1 & \beta^i\\0 & 1\end{pmatrix} = \left(\infty, (\beta^i + h)\right) = (c, d)$$

Thus $c = \infty$ and $d = \beta^i + h$. ■

6.3.4.2 Example

Using Theorem 6.3.4.1 we can construct suborbital graph $\Gamma_{(\infty,\beta^i)}$ for $PGL(2,3)$ corresponding to suborbit Δ_2 (in Example 6.3.1.2)

Figure 6.3.4.1 Suborbital graph formed by pairs of the form $(\infty, \boldsymbol{\beta^i})$



6.3.5 Suborbital graph corresponding to suborbit of G formed by pairs of the form $(\mathbf{1}, \boldsymbol{\beta}^{\boldsymbol{i}})$

6.3.5.1 Theorem

For each of the following cases $((v,h),(c,d))$ is an edge in $\Gamma_{(1,\beta^i)}$.

a) $v,h \neq \infty$, $c = (v+h)(2)^{-1}$ and $d = (v\beta^i + h)(\beta^i+1)^{-1}$

b) $h = \infty$, $c = (v+1)$and $d = (v\beta^i + 1)\beta^{-i}$

c) $v = \infty$, $c = 1 + h$ and $d = \beta^i + h$.

Proof

a) $v,h \neq \infty$, so we take $g \in G$ to be $\begin{pmatrix} v & h \\ 1 & 1 \end{pmatrix}$; so

$$g(\infty, \beta^i) = (c,d)$$

Hence;

$$\begin{pmatrix} v & h \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & \beta^i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} v+h & v\beta^i + h \\ 2 & \beta^i + 1 \end{pmatrix}$$

$$= (\ (v+h)(2)^{-1}, (v\beta^i + h)(\beta^i+1)^{-1})$$

Thus $c = (v+h)(2)^{-1}$ $\quad d = (v\beta^i + h)(\beta^i+1)^{-1}$.

b) Since $h = \infty$, In this case we take $g \in G$ to be $\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix}$, so we have;

$$g(1, \beta^i) = (c,d)$$

$$\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \beta^i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} v+1 & \beta^i + 1 \\ 1 & \beta^i \end{pmatrix}$$

So $c = v + 1$ and $d = (\beta^i + 1)\beta^{-i}$

c) For $v = \infty$, then we have,

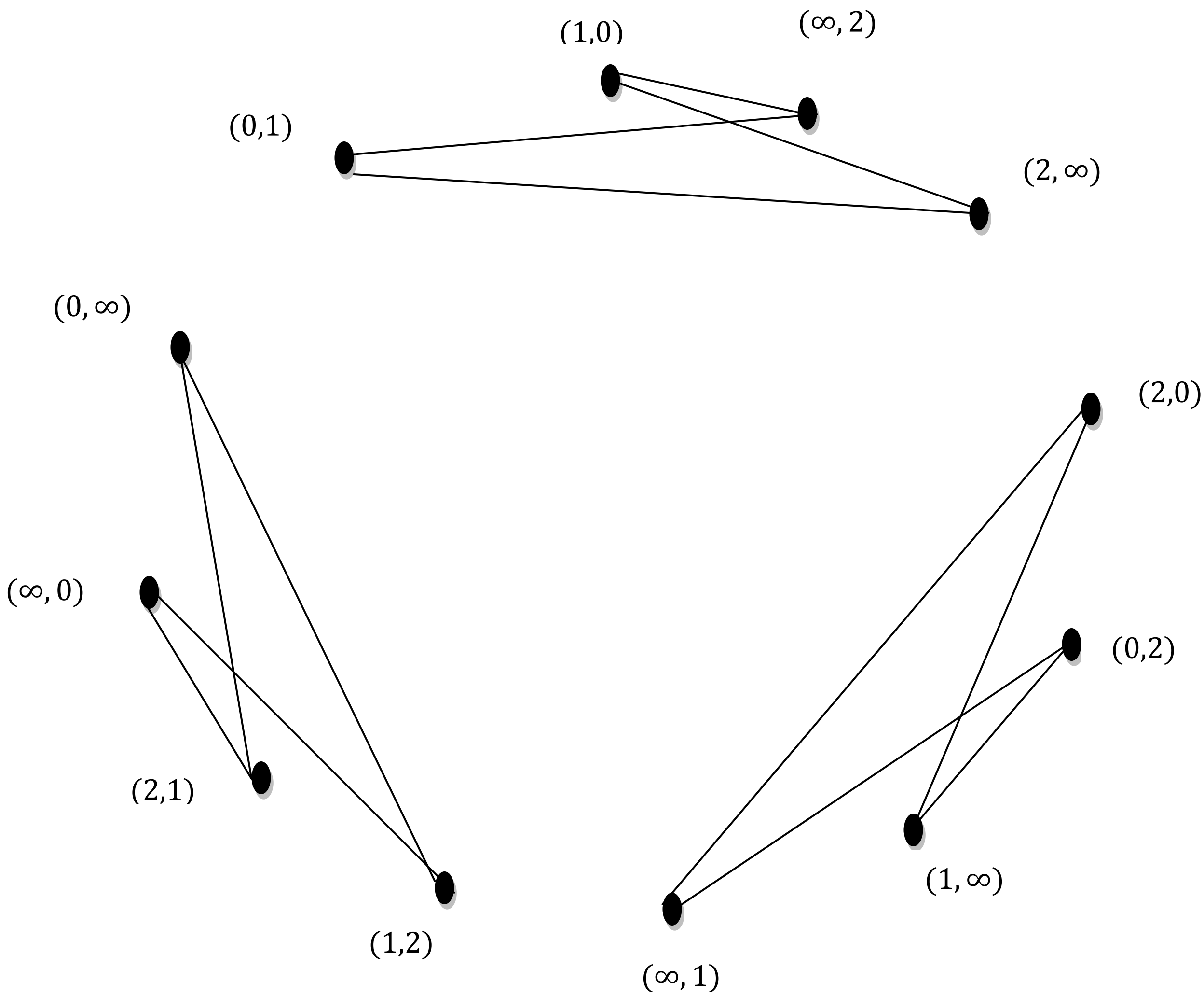
$$\begin{pmatrix}1 & h\\0 & 1\end{pmatrix}\begin{pmatrix}1 & \beta^i\\1 & 1\end{pmatrix} = ((1+h),(\beta^i + h)) = (c,d)$$

Hence $c = 1 + h$ and $d = \beta^i + h$ ■

6.3.5.2 Example

Using Theorem 6.3.5.1 we can construct suborbital graph $\Gamma_{(1,\beta^i)}$ for $PGL(2,3)$ corresponding to suborbit Δ_6 (in Example 6.3.1.2)

Figure 6.3.5.1: Suborbital graph formed by pairs of the form $(\mathbf{1}, \boldsymbol{\beta}^{\boldsymbol{i}})$



6.3.6 Suborbital graph corresponding to suborbit of G

formed by the pairs $(\mathbf{0}, \infty)$

6.3.6.1 Theorem

For each of the following cases $((v, h), (c, d))$ is an edge in $\Gamma_{(0,\infty)}$.

a) $v, h \neq \infty$, $c = h$ and $d =$ v

b) $c = h$ and $d =$ v $= \infty$

c) $c = h = \infty$ and $d = v$

Proof

a) Since $v, h \neq \infty$, then we take $g \in G$ to be $\begin{pmatrix} v & h \\ 1 & 1 \end{pmatrix}$. So we have;

$$g(0, \infty\ ) = (c, d).$$

Hence;

$$\begin{pmatrix} v & h \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} h & v \\ 1 & 1 \end{pmatrix} = (\ h, \mathrm{v})$$

Thus $c = h \quad d =$ v.

b) For $v = \infty$, so taking $g \in G$ to be $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$, so we have;

$$g(0, \infty\ ) = (c, d)$$

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} h & 1 \\ 1 & 0 \end{pmatrix}.$$

So $c = h$ and $d = \infty$.

c) Finally for $h = \infty$, then we have,

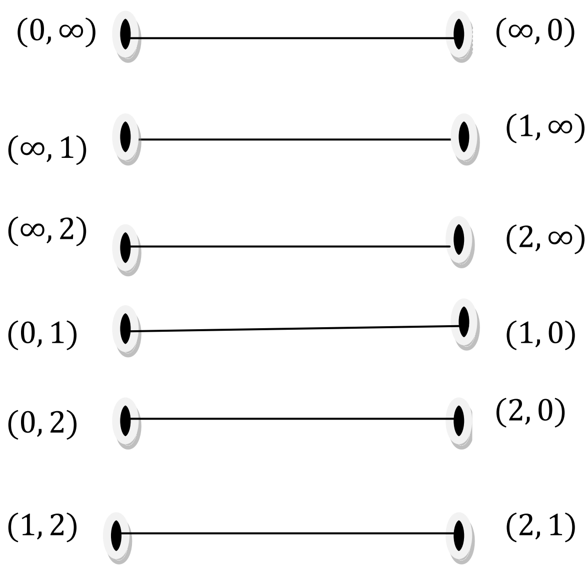
$$\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\infty, \mathrm{v}) = (c, d).$$

Thus $c = \infty$ and $d =$ v. ■

6.3.6.2 Example

Using Theorem 6.3.6.1 we can construct suborbital graph $\Gamma_{(0,\infty)}$ for $PGL(2,3)$ corresponding to suborbit Δ_2 (in Example 6.3.1.2)

Figure 6.3.6.1: Suborbital graph formed by pairs of the form $(\mathbf{0},\infty)$



6.4 properties of the suborbital graphs constructed

6.4.1 Theorem

The suborbital graphs $\Gamma_{(1,0)}, \Gamma_{(0,1)}, \Gamma_{(\infty,\beta^i)}, \Gamma_{(\beta^i,\infty)}$ are of girth 3.

Proof

For $\Gamma_{(\infty,\beta^i)}$

Since $(\infty, 1), (\infty, \beta) \in \Delta_2$, in $\Gamma_{(\infty,\beta^i)}$, $(\infty, 0)$ is adjacent to $(\infty, 1)$ and (∞, β). By Theorem 6.3.4.1 and taking (v, h) to be $(\infty, \beta)$and β^i to be $\beta - 1$ we find that $((\infty, 1), (\infty, \boldsymbol{\beta}))$ is an edge in $\Gamma_{(\infty,\beta^i)}$ giving a triangle. ■

NB: Applying the same approach to the other suborbital graphs we obtain the results.

6.4.2 Theorem

The suborbital graph $\Gamma_{(0,\infty)}$ is of girth 0. ■

Proof

The graph is regular of degree one hence cannot form a circle.

6.4.3 Theorem

The suborbital graphs $\Gamma_{(1,\beta^i)}$ is of girth 4.

Proof

Since $(1, \boldsymbol{\beta}), (\boldsymbol{\beta}, 1) \in \Delta_6$, in $\Gamma_{(1,\beta^i)}$, $(\infty, 0)$is adjacent to $(1, \boldsymbol{\beta})$ and $(\boldsymbol{\beta}, 1)$. By Lemma 6.1.8 and Theorem 6.3.5.1 and taking (v, h) to be $(1, \boldsymbol{\beta})$and β^i to be -1 we find that $((1, \boldsymbol{\beta}), (0, \infty))$ is an edge**.** Also applying the same Theorem and taking (v, h) to be $(0, \infty)$ we find that $((0, \infty), (\boldsymbol{\beta}, 1))$ is an edge in $\Gamma_{(1,\beta^i)}$ giving us girth 4.

6.4.4 Theorem

The number of connected components for $\Gamma_{(0,\infty)}$ is $\frac{q(q+1)}{2}$.

Proof

The number of ordered pairs is $\binom{q+1}{2}$. Each connected components has two ordered pairs. Hence the results follow.

CHAPTER SEVEN

CONCLUSION AND RECOMMENTATIONS

This chapter is divided into two sections. In section 7.1 it gives the conclusion. Section 7.2 gives recommendation for further research.

7.1 Conclusion

The purpose of this research was mainly to investigate the action of $PGL(2, q)$ on the cosets of its subgroups namely; $C_{q-1}, C_{q+1}, P_q, A_4, A_5$ and $D_{2(q-1)}$. Corresponding to each action the disjoint cycle structures, cycle index formulas, ranks and the subdegrees were computed. Suborbital graphs corresponding to the action of $PGL(2, q)$ acting on the cosets of C_{q-1} were constructed.

To obtain these, some objectives were set which we achieved as follows;

a) For the disjoint cycle structures of $PGL(2, q)$ on the cosets of its subgroups namely; $C_{q-1}, C_{q+1}, P_q, A_4, A_5$ and $D_{2(q-1)}$the results are given in chapter three.

b) The cycle index formulas for the representation of $PGL(2, q)$ on the cosets of its subgroups were computed in chapter four.

c) In chapter five we determine the ranks and the subdgrees for the action of $PGL(2, q)$ on the cosets of its subgroups. Results for the ranks are in Section 5.1. For the subdegrees they are section 5.2.

d) In Chapter six suborbital graphs Γ of G corresponding to its suborbits when G acts on the cosets of $H = C_{q-1}$ are constructed and their theoretic properties studied. All suborbital graphs except two are self paired. We also established that some were of girth 3 others were not.

7.2 Recommendation for further research

Having determined the disjoint cycle structures, cycle index formulae, ranks, subdegrees and constructing suborbital graphs when $PGL(2,q)$ acts on the cosets of C_{q-1}, one can extend this work by constructing suborbital graphs of $PGL(2,q)$on the cosets of $C_{q+1}, P_q, A_4\ and\ A_5$.

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