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On Modelling and Pricing Rainfall Derivatives with Seasonality

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ABSTRACT *We are interested in pricing rainfall options written on precipitation at specific locations. We assume the existence of a tradeable financial instrument in the market whose price process is affected by the quantity of rainfall. We then construct a suitable 'Markovian gamma' model for the rainfall process which accounts for the seasonal change of precipitation and show how maximum likelihood estimators can be obtained for its parameters.*

We derive optimal strategies for exponential utility from terminal wealth and determine the utility indifference price of the claim. The method is illustrated with actual measured data on rainfall from a location in Kenya and spot prices of Kenyan electricity companies.

KEY WORDS: Rainfall derivatives, Seasonality, Discrete-time Markov control process, Utility indifference pricing, Monte Carlo methods
MATHEMATICS SUBJECT CLASSIFICATION (2000): 49L20, 60J10, 65C05

1. Introduction

A developing country's economy is vulnerable to extreme climatic changes with disastrous consequence. For instance, Kenya is susceptible to severe drought and seasonal floods.

An increased understanding of both the dynamics driving the natural disasters and the possible hedging mechanisms would offer a new way to transfer financial risks originating in the uncertainties of weather and climate to financial markets, which has wide risk management strategies. Unlike independent events, such as fires or auto accidents, weather-related catastrophes affect a large proportion of people within a single area, resulting in highly correlated losses which translate to a larger than expected number of insurance claims. This leads to constraints on the developing country's economy, as most developing countries have a small number of insurance companies or no insurance cover for weather-related events at all. The weather derivatives not only create a new opportunity for dealing with catastrophic or disaster risk but also act as a new private-based insurance product to highly

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weather-dependent sectors such as agricultural industries (e.g. see Zapranis and Alexandridis, 2008) and hydro-power electric generating companies. These derivatives are mainly not derived from the exchange-traded securities and can be defined as a contract between two parties that states how payment will be exchanged between the parties depending on certain meteorological conditions during the contract period.

In our study, we consider a discrete-time Markovian model with the main objective of being able to account for rainfall seasonality as a factor in pricing rainfall derivatives. The model is such that precipitation in every fixed month of the year has gamma distribution, which is chosen because it fits the data well and it is possible to estimate the parameters using maximum likelihood estimation (MLE). We then model a financial instrument whose price process dynamics are correlated to precipitation. This could be forward contracts on electric power in a country that mainly depend on hydro-power or the price of a share of a company whose business depends on rainfall or lack of it. This enables an investor to hedge the risk of a rainfall derivative in the market. In particular, one can compute the derivative's utility indifference price.

A similar view was shared by Carmona and Diko (2005), who proposed a time-homogeneous jump Markov process to model the rainfall process. They assumed the rainfall process to be composed of storms which are in turn composed of rainfall cells. At a cell arrival time, the rainfall process jumps up by a random amount and at the cell's extinction time, it jumps down by a random amount with appropriate distribution. They used MLE to fit the model to Norwegian data. Then they gave indifference prices of certain precipitation-based options for exponential utility and bounds on the price for power utility. Despite being very elegant, this precipitation model has the disadvantage of not taking into account seasonal variation of rainfall. For some countries this disadvantage may even render that model useless.

In this study, we develop a Markovian gamma model incorporating the seasonality effect for the rainfall process of the region under study, which is therefore not time-homogeneous. The model is crafted in a way that makes the precipitation within a period, for example 1 month, gamma-distributed. The popularity of the gamma distribution for describing precipitation data is derived from the fact that it provides a flexible representation and is widely used to represent precipitation (e.g. see Thom, 1958). We then describe how to maximum likelihood estimate the parameters of our model to fit it to historical precipitation data.

Implicit in our model is the assumption that precipitation can be forecast with sufficient accuracy for the period corresponding to one time step in our model. As we use monthly precipitation data to calibrate our model and therefore the time step is 1 month, this assumption seems overly optimistic. However, one could do the same for weekly data, and then the assumption of perfect forecast is rather sound.

This article is organized as follows. In Section 2, we construct the Markovian gamma model, then discuss how to estimate the parameters of the model in Section 3. In Section 4 we present our main result; we derive the pricing formula for the rainfall derivative. Section 5 deals with the numerical implementation of the model using historical precipitation data from Kenya and an electricity company listed at the Kenyan stock exchange. Finally, we discuss the results of our study in Section 6.

2. Model Setup

2.1 Markovian Precipitation Model with Seasonality

To account for the seasonal variation of precipitation over the year we partition the year into m periods of equal length and model the total amount of rainfall within one period separately.

Let $Y_0, Y_1, \ldots$ be the sequence of total precipitation per period. We assume that in period k the precipitation has cumulative distribution function (CDF) $F_{k \bmod m}$, $k \geq 0$. We further assume that F_k is continuous and strictly increasing, such that the inverse F_k^{-1} exists and is strictly increasing and continuous.

From these assumptions it follows that the sequence $(F_{k \bmod m}(Y_k))_{k \geq 0}$ consists of – generally dependent – random variables with uniform distribution on (0, 1).

On the other hand, from a sequence of (0, 1)-uniform random variables $U_0, U_1, \ldots$ we can generate a sample path of future precipitation by setting $Y_k := F^{-1}_{k \bmod m}(U_k)$, $k \geq 0$, using the standard inverse transform method (e.g. see Glasserman, 2004).

Moreover, as the amounts of precipitation of two consecutive months are hardly independent, we will propose some correlation structure as follows:

Assumption 2.1. The sequence $(F_{k \bmod m}(Y_k))$ is a discrete-time Markov process with state space (0, 1).

Obviously there are infinitely many possible candidates to use in a precipitation model. We give an example which we will use later in our numerics and which has the advantage that there is only one parameter and that this parameter can be estimated using the MLE method.

Example 2.1 (Gaussian copula). Let $U_0, U_1, \ldots$ be a sequence of independent random variables, uniform on (0, 1). Define

$$\widehat{U}_0 = U_0,$$
$$\widehat{U}_{k+1} = \Phi\Big(\rho\Phi^{-1}(\widehat{U}_k) + \sqrt{1-\rho^2}\Phi^{-1}(\widehat{U}_{k+1})\Big) \quad \text{for } k > 0.$$

Then $(\widehat{U}_k)_{k \geq 0}$ is a discrete-time, time-homogeneous Markov process with state space (0, 1).

Here and in the rest of this article, Φ denotes the cumulative probability distribution function of a standard normal random variable.

Remark 2.1. Note that the process $(\widehat{U}_k)_{k \geq 0}$ in Example 2.1 admits a certain monotonicity property: a shift in $\widehat{U}_0$ results in a shift of the whole path of the chain in the same direction.

In the numerical example we will assume that precipitation within period k has gamma distribution with shape parameter α_k and scale parameter β_k, that is $F_k(z) = \int_{-\infty}^{z} f(y)\mathrm{d}y$ where

$$f_k(y) = \begin{cases} \frac{1}{\beta_k \Gamma(\alpha_k)} \left(\frac{y}{\beta_k}\right)^{\alpha_k - 1} \mathrm{e}^{-y/\beta_k} & \text{if } y \geq 0, \alpha_k, \beta_k > 0, \\ 0 & \text{if } y < 0. \end{cases} \tag{1}$$

2.2 Market Model

Let Y_n denote the precipitation in period $[n, n+1]$ and suppose that this is known via forecasts at the beginning of each period. We model the dynamics of the price process of an asset S as follows: Let S_0 be some fixed real number and

$$S_{n+1} = S_n + \mu(Y_n) + \sigma(Y_n) Z_{n+1}, \tag{2}$$

where $Z_1, \ldots, Z_N$ are independent standard normal variables and μ and σ are measurable functions, σ is bounded away from 0. $Z_1, \ldots, Z_N$ are assumed to be independent of $Y_0, \ldots, Y_{N-1}$. Throughout this article we assume zero interest.

The concrete form of μ and σ has yet to be specified. In our numerical examples, we shall assume that σ is constant and that μ is of the form

$$\mu(y) = a \log(\varepsilon + y) + b. \tag{3}$$

This form is the same as was used by Carmona and Diko (2005) for a similar model. The purpose of $\varepsilon > 0$ is to prevent the argument of log to become zero. We will show in Section 3.3 how to estimate the parameters a and b.

3. Parameter Estimations

Suppose we are given historical data $y_0, \ldots, y_{n-1}$ of precipitation at a specific location with m observations per year. We want to fit our proposed model to the actual data, that is, we want to find a set of parameters $\rho, \alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{m-1}$ such that, if F_k is the CDF of a gamma distribution with parameters α_k, β_k for each k and the underlying Markov process is of the form described in Example 2.1, then $y_0, \ldots, y_{n-1}$ has maximum likelihood. For example, monthly observations would give $m = 12$.

We have to make some concessions here in that we do not find a joint maximum likelihood estimator for the whole set of parameters.

Instead, we first estimate (α_k, β_k) for every month using ordinary MLE for this (Figure 1). Strictly speaking this is not allowed because in the MLE procedure we assume that the given data are independent, which is not the case in our model. However, though we cannot and do not want to ignore correlation between consecutive months, we do not expect to make big error by assuming that consecutive Januaries are independent.

Having found the CDFs $F_0, \ldots, F_{m-1}$ in this way we then may compute $z_k := \Phi^{-1}(F_{k \bmod m}(y_k))$. From this sequence of (roughly) standard normal variables we can estimate the auto-correlation ρ using MLE. We have to remark here that strictly speaking this procedure is not correct: the precipitation data for two consecutive Januaries, say, are not independent samples from the same distribution in our model

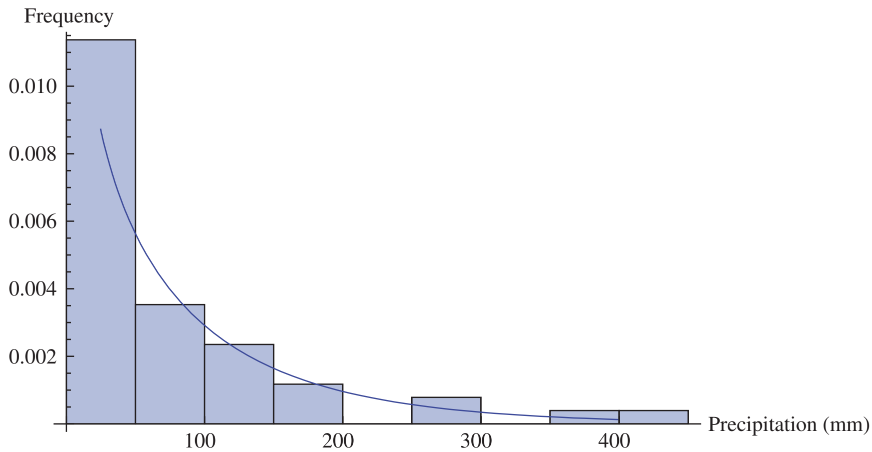

Figure 1. January rain, Dagoretti weather station (gamma distribution with $\alpha = 0.458$ and $\beta = 159.469$). The cumulative monthly precipitation is selected from the Kenya Meteorological Department for the period 1955–2005 with roughly 6% of the data censored, that is equal to 0.

(Figure 1). It is, however, fair to assume that they are almost independent, because – at least when one uses the Gaussian copula for modelling correlation – the correlation between two non-consecutive time periods decreases rapidly with the lag.

In the next section we discuss how to use MLE of parameters for the gamma distribution for data containing zeros.

3.1 MLE for the Gamma Distribution Using Data Containing Zeros

The following method is taken from Wilks (1990).

Suppose the given data set contains M_0 data points recorded as zeros. These are interpreted as censored points, where the censoring level[1] is A and points M_v with known values where $M = M_0 + M_v$. The likelihood function for the distribution parameters is given by

$$Y(\alpha, \beta; y) = \prod_{j=1}^{M_0} G(A; \alpha, \beta) \prod_{i=1}^{M_v} g(y_i; \alpha, \beta)$$
$$= [G(A; \alpha, \beta)]^{M_0} \prod_{i=1}^{M_v} \frac{1}{\beta \Gamma(\alpha)} \left(\frac{y_i}{\beta}\right)^{\alpha - 1} \mathrm{e}^{-y_i/\beta},$$

where

$$G(A; \alpha, \beta) = \int_0^A g(y_j; \alpha, \beta) \mathrm{d}y = \mathbb{P}[y_j \leq A].$$

Consider first $M_0 = 0$, that is all the data values are known. Then it is easily shown that the maximum likelihood estimators of the parameters satisfy

$$\log(\beta) + \eta(\alpha) = \sum_{i=1}^{M_v} \frac{\log\ y_i}{M_v},$$

$$\alpha - \frac{1}{\beta}\sum_{i=1}^{M_v} \frac{y_i}{M_v} = 0,$$

where $\eta(\alpha) = \frac{\partial \log[\Gamma(\alpha)]}{\partial \alpha}$ is the digamma function. Depending on $\bar{y} = \sum_{i=1}^{M_v} y_i / M_v$ we obtain

$$\log\left(\frac{1}{\alpha}\bar{y}\right) + \eta(\alpha) = \frac{1}{M_v}\sum_{i=1}^{M_v} \log(y_i).$$

Hence, the MLE satisfies $\log(\widehat{\alpha}) - \eta(\widehat{\alpha}) = \log(\overline{y}) - \frac{1}{M_v}\sum_{i=1}^{M_v} \log(y_i)$ and $\widehat{\beta} = \frac{\overline{y}}{\widehat{\alpha}}$. $\widehat{\alpha}$ and $\widehat{\beta}$ can thus be computed numerically.

In the case $M_0 \neq 0$, the maximization of

$$L(\alpha, \beta; y) = M_0 \log[G(A; \alpha, \beta)] - M_v[\alpha \log(\beta) + \log(\Gamma(\alpha))] + (\alpha - 1)\sum_{i=1}^{M_v} \log(y_i) - \frac{1}{\beta}\sum_{i=1}^{M_v} y_i$$

can be computed numerically for the values of α and β, for example using Mathematica.

3.2 MLE of the Correlation Coefficient

Suppose we want to estimate the correlation coefficient parameter ρ, in Example 2.1. We let

$$z_0 = w_0,$$
$$z_{k+1} = \rho z_k + \sqrt{1 - \rho^2} w_{k+1}, \text{ hence } w_{k+1} = \frac{z_{k+1} - \rho z_k}{\sqrt{1 - \rho^2}},$$

where $z_0, \ldots, z_{n-1}$ are as before, that is $z_k = \Phi^{-1}(F_{k \bmod m}(y_k))$. According to our assumptions, $w_0, \ldots, w_{n-1}$ should be standard normal. Then it is straightforward to verify that the maximum likelihood estimate of the parameter is given by

$$\rho = \begin{cases} 0 & \text{if } \sum_{k=1}^{n-1} z_{k-1} z_k = 0, \\ b + \sqrt{b^2 - 1} & \text{if } \sum_{k=1}^{n-1} z_{k-1} z_k < 0, \\ b - \sqrt{b^2 - 1} & \text{if } \sum_{k=1}^{n-1} z_{k-1} z_k > 0, \end{cases} \quad \text{where } b = \frac{\sum_{k=1}^{n-1} (z_{k-1}^2 + z_k^2)}{2\sum_{k=1}^{n-1} z_{k-1} z_k}.$$

3.3 Estimating the Parameters for μ

As mentioned earlier we want to model μ as

$$\mu(y) = a \log(\varepsilon + y) + b \,.$$

We set $\varepsilon = 0.01$. The exact value of ε does not make much difference as long as it is small compared to y.

a and b can then be estimated from the combined rainfall and market data via MLE: given precipitation records $y_0, \dots, y_{n-1}$ and asset prices $s_0, \dots, s_n$ set $x_k := \log(\varepsilon + y_k)$ and $\Delta_k := s_k - s_{k-1}$. Then

$$a x_k + b + \sigma Z_{k+1} = \Delta_{k+1} \,,$$

where $Z_1, \dots, Z_n$ are assumed to be independent standard normal variables.

It is not hard to show that the maximum likelihood estimates for the parameters a, b, σ are given by

$$a = \frac{\varepsilon - \alpha\gamma}{\gamma^2 - \delta} \,, \quad b = \frac{\gamma\varepsilon - \alpha\delta}{\gamma^2 - \delta} \,,$$
$$\sigma^2 = \beta + b^2 + a^2\delta - 2b\alpha - 2a\varepsilon + 2ab\gamma,$$

where

$$\alpha = \frac{1}{n}\sum_{k=0}^{n-1} \Delta_{k+1}, \quad \beta = \frac{1}{n}\sum_{k=0}^{n-1} \Delta_{k+1}^2, \quad \gamma = \frac{1}{n}\sum_{k=0}^{n-1} x_k, \quad \delta = \frac{1}{n}\sum_{k=0}^{n-1} x_k^2, \quad \varepsilon = \frac{1}{n}\sum_{k=0}^{n-1} \Delta_{k+1} x_k.$$

4. Indifference Pricing of Rainfall Options

4.1 Utility Maximization Without a Derivative

In this section we study the existence of an optimal strategy for maximizing expected utility under our model at the end of a finite trading period in a financial market.

Here we assume that trading can only take place at discrete time intervals and that precipitation is known one period ahead via forecasts.

We assume the market model presented in Section 2.2, that is that the dynamics are given by

$$S_{n+1} = S_n + \mu(Y_n) + \sigma(Y_n) Z_{n+1} \,, \tag{4}$$

where $Z_1, \dots, Z_N$ are independent standard normal variables and μ and σ are measurable functions, σ is bounded away from 0. $Z_1, \dots, Z_N$ are assumed to be independent of $Y_0, \dots, Y_{N-1}$ and Y is a Markov process.

Let $\theta = \{\theta_n\}_{0 \le n \le N-1}$ be a trading strategy which describes the investor's portfolio as carried forward for a finite time. Particularly, θ_n is the number of units of the security (e.g. a share) held between n and $n+1$ and θ_n may depend only on information available at time n.

Given a strategy θ, the dynamic of the wealth process $X^\theta = (X_n^\theta)_{n=0}^N$ is described by

$$X_{n+1}^\theta = X_n^\theta + \theta_n(S_{n+1} - S_n), \quad n \geq 0. \tag{5}$$

We assume that the agents' preferences are described by an exponential utility function $U(x) = -\exp(-\alpha x)$ for some $\alpha > 0$. We therefore seek a strategy θ which maximizes the expected utility over N stages for a given initial wealth x;

$$\sup_\theta \mathbb{E}[U(X_N^\theta))|X_0^\theta = x]. \tag{6}$$

Definition 4.1. For all $x \in \mathbb{R}$ and $y \in [0, \infty)$ define

$$V_N(x,y) := U(x)$$

and

$$V_n(x,y) := \sup_{\theta_n,\ldots,\theta_{N-1}} \mathbb{E}[U(X_N^\theta)|X_n^\theta = x,\, Y_n = y] \quad \text{for } 0 \leq n < N.$$

Note that $V_n(x,y) < \infty$ for all n as U is bounded from above. We also have $V_n(x,y) > -\infty$, as $\theta \equiv 0$ gives a finite value. The quantity $V_n(x,y)$ is the highest terminal expected utility for an agent who starts trading at time n with initial endowment x and who knows that the precipitation for the coming period will be equal to y.

Proposition 4.1. Under the above assumptions the optimal trading strategy θ^* is given by

$$\theta_n^* = \frac{\mu(Y_n)}{\alpha\sigma(Y_n)^2}$$

and the optimal value at n is given by

$$V_n(x,y) = -\mathrm{e}^{-\alpha x}\mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{k=n}^{N-1}\frac{\mu^2(Y_k)}{\sigma^2(Y_k)}\right)|\, Y_n = y\right].$$

Proof. Note that trivially V_N has the claimed form. Consider now n with $n < N$. Write $b_n(y) = \mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{k=n}^{N-1}\frac{\mu^2(Y_k)}{\sigma^2(Y_k)}\right)|\, Y_n = y\right]$ and $\Delta S_{n+1} := \mu(Y_n) + \sigma(Y_n)Z_{n+1}$:

$$
\begin{aligned}
V_n(x,y) &= \sup_{\theta_n,\ldots,\theta_{N-1}} \mathbb{E}\left[U\left(X_n + \sum_{k=n}^{N-1}\theta_k \Delta S_{k+1}\right) | X_n = x,\, Y_n = y\right] \\
&= U(x) \inf_{\theta_n,\ldots,\theta_{N-1}} \mathbb{E}\left[\mathrm{e}^{-\alpha \sum_{k=n}^{N-1}\theta_k \Delta S_{k+1}} \,|\, Y_n = y\right],
\end{aligned}
$$

as U is exponential utility. It remains to show that the second factor equals $b_n(y)$:

$$
\begin{aligned}
c_n(y) &:= \inf_{\theta_n,\ldots,\theta_{N-1}} \mathbb{E}\left[\mathrm{e}^{-\alpha \sum_{k=n}^{N-1}\theta_k \Delta S_{k+1}} \,|\, Y_n = y\right] \\
&\overset{(a)}{=} \inf_{\theta_n} \mathbb{E}\left[\mathrm{e}^{-\alpha\theta_n \Delta S_{n+1}} \inf_{\theta_{n+1},\ldots,\theta_{N-1}} \mathbb{E}\left[\mathrm{e}^{-\alpha \sum_{k=n+1}^{N-1}\theta_k \Delta S_{k+1}} \,|\, Y_{n+1}, Z_{n+1}\right] | \, Y_n = y\right] \\
&\overset{(b)}{=} \inf_{\theta_n} \mathbb{E}\left[\mathrm{e}^{-\alpha\theta_n \Delta S_{n+1}} \inf_{\theta_{n+1},\ldots,\theta_{N-1}} \mathbb{E}\left[\mathrm{e}^{-\alpha \sum_{k=n+1}^{N-1}\theta_k \Delta S_{k+1}} \,|\, Y_{n+1}\right] | \, Y_n = y\right] \\
&= \inf_{\theta_n} \mathbb{E}\left[\mathrm{e}^{-\alpha\theta_n \Delta S_{n+1}} b_{n+1}(Y_{n+1}) | \, Y_n = y\right] \\
&\overset{(c)}{=} \inf_{\theta_n} \mathbb{E}\left[\mathrm{e}^{-\alpha\theta_n(\mu(y)+\sigma(y)Z_{n+1})} | \, Y_n = y\right] \mathbb{E}[b_{n+1}(Y_{n+1}) | \, Y_n = y] \\
&= \mathbb{E}[b_{n+1}(Y_{n+1}) | \, Y_n = y] \inf_{\theta_n} \mathbb{E}\left[\mathrm{e}^{-\alpha\theta_n(\mu(y)+\sigma(y)Z_{n+1})}\right].
\end{aligned}
$$

Here (a) holds because $\theta_{n+1}, \ldots, \theta_{N-1}$ are allowed to depend on Z_{n+1}; (b) holds because $\sum_{k=n+1}^{N-1} \theta_k \Delta S_{k+1}$ and Z_{n+1} are independent by construction; (c) holds because Z_{n+1} and $Y_n, \ldots, Y_{N-1}$ are independent and therefore are also independent conditional on $Y_n = y$. We can perform the maximization with respect to θ_n to get the optimal control:

$$
\theta_n^* = \frac{\mu(y)}{\alpha\sigma(y)^2}. \tag{7}
$$

Substituting back gives us

$$
\begin{aligned}
c_n(y) &= \mathbb{E}[b_{n+1}(Y_{n+1}) | \, Y_n = y] \exp\left(-\frac{1}{2}\frac{\mu(y)^2}{\sigma(y)^2}\right) \\
&= \mathbb{E}\left[b_{n+1}(Y_{n+1}) \exp\left(-\frac{1}{2}\frac{\mu(Y_n)^2}{\sigma(Y_n)^2}\right) | \, Y_n = y\right] \;=\; b_n(y)
\end{aligned}
$$

as required. □

4.2 Utility Maximization with a Derivative

We now consider the optimization problem for an agent who holds a derivative written on precipitation. The nature of the options under consideration is the same as the ones considered by Carmona and Diko (2005).

It is assumed that the payoff of the option is of the form

$$H(Y) := \left(\sum_{k=n_1}^{n_2} h(Y_k) - K \right)^+ ,$$

where h is some non-negative function. Special cases which are of interest are

(1) $h(y) = y$, that is the contract specifies that its holder receives the difference between the cumulative rainfall during periods n_1 to n_2, and K, if this difference is positive, and zero otherwise;
(2) $h(y) = 1_{y>c}$, that is the holder of the option gets the difference between the number of periods between n_1 and n_2 for which rainfall was above level c, and K, if this difference is positive, and zero otherwise.

We assume $K = 0$ (the result can be extended to the case where $K > 0$ is analogous to the method in Carmona and Diko (2005)), that is

$$\left(\sum_{n=n_1}^{n_2} h(Y_n) - K \right)^+ = \sum_{k=n_1}^{n_2} h(Y_n) = \sum_{n=0}^{N-1} h(Y_n) 1_{\{n_1,\ldots,n_2\}}(n) =: \sum_{n=0}^{N-1} g(n, Y_n).$$

The dynamic of the wealth process of an investor following strategy ϕ is given by

$$X_{n+1}^{\phi} = X_n^{\phi} + \phi_n(\mu(Y_n) + \sigma(Y_n) Z_{n+1}).$$

Our objective is to compute

$$\sup_{\phi} \mathbb{E}\left[U\left(X_N^{\phi} + \sum_{k=0}^{N-1} g(k, Y_k) \right) | X_0^{\phi} = x, Y_0 = y \right].$$

This is equivalent to computing

$$\sup_{\phi} \mathbb{E}\left[U(\tilde{X}_N^{\phi}) | X_0^{\phi} = x, Y_0 = y \right],$$

where the dynamic of $\tilde{X}^{\phi}$ is described by

$$\tilde{X}_{n+1}^{\phi} = \tilde{X}_n^{\phi} + g(n, Y_n) + \phi_n(\mu(Y_n) + \sigma(Y_n) Z_{n+1}).$$

Now similar calculations as before yield the following result:

Proposition 4.2. Under the above assumptions the optimal trading strategy ϕ^* for an investor holding a derivative with payoff $H(Y) = \sum_{n=0}^{N-1} g(n, Y_n)$, $g \geq 0$, is given by

$$\phi_n^* = \frac{\mu(Y_n)}{\alpha\sigma(Y_n)^2}$$

and the optimal value at n is given by

$$V_n^d(x, y) = -\mathrm{e}^{-\alpha x}\mathbb{E}\left[\exp\left(-\sum_{k=n}^{N-1}\left(\frac{1}{2}\frac{\mu^2(Y_k)}{\sigma^2(Y_k)} + \alpha g(k, Y_k)\right)\right) | Y_n = y\right].$$

So far we have only considered the buyer's position. The situation of the seller of the derivative is similar and can be treated nearly the same way with the objective

$$\sup_{\phi} \mathbb{E}\left[U\left(X_N^{\phi} - \sum_{k=0}^{N-1} g(k, Y_k)\right) | X_0^{\phi} = x, Y_0 = y\right].$$

Note that now it can happen that for some (or even every) strategy ϕ we have $\mathbb{E}[U(\tilde{X}_N^{\phi})|X_0^{\phi} = x, Y_0 = y] = -\infty$. To avoid this we make the following assumption.

Assumption 4.1. For all $n = 0, \ldots, N-1$ and all y we have

$$\mathbb{E}\left[U\left(\sum_{k=n}^{N-1} g(k, Y_k)\right) | Y_n = y\right] > -\infty.$$

This assumption guarantees that the optimal value functions for the seller are finite valued. We have the following result whose proof is analogous to those of Propositions 4.1 and 4.2.

Proposition 4.3. Under the above assumptions the optimal trading strategy ϕ^* for an investor being short a derivative with payoff $H(Y) = \sum_{n=0}^{N-1} g(n, Y_n)$, $g \geq 0$, is given by

$$\theta_n^* = \frac{\mu(Y_n)}{\alpha\sigma(Y_n)^2}$$

and the optimal value at n is given by

$$V_n^d(x, y) = -\mathrm{e}^{-\alpha x}\mathbb{E}\left[\exp\left(-\sum_{k=n}^{N-1}\left(\frac{1}{2}\frac{\mu^2(Y_k)}{\sigma^2(Y_k)} - \alpha g(k, Y_k)\right)\right) | Y_n = y\right].$$

Note that in our setup the investor follows the same optimal strategy

$$\phi_n = \frac{\mu(Y_n)}{\alpha\sigma^2(Y_n)}$$

regardless of whether he/she holds the derivative or not. This strange feature also exists in Carmona and Diko (2005).

Benth and Proske (2009) analysed the effectivity of the hedging strategy induced by the indifference pricing paradigm in the context of interest rate guarantees. This hedging strategy is defined as the difference between the optimal strategy of an investor not holding the derivative and the optimal strategy of an investor holding the derivative and having the same initial wealth, diminished by the indifference price. It is then interesting to ask how effectively the risk of the derivative is reduced by following this hedging strategy.

However, in the present setup this hedging strategy is zero. Nevertheless we adopt the terms 'hedged' and 'unhedged,' referring to the circumstance that an investor with access to a market admitting an asset which is correlated to precipitation is exposed to less risk when buying the derivatives than an investor without that capability. That means that there is no straightforward meaningful way to extend the notion of effectivity of the hedging strategy from Benth and Proske (2009) to our problem.

4.3 The Indifference Price

We recall that our wealth process without derivative is

$$X^{\theta}_{n+1} = X^{\theta}_n + \theta_n(\mu(Y_n) + \sigma(Y_n)Z_{n+1}),$$

let the value function of our investments without rainfall derivatives at time 0 be

$$V(x,y) = \sup_{\theta} \mathbb{E}[U(X^{\theta}_N)|X^{\theta*}_0 = x,\, Y_0 = y].$$

We have shown that there exists a strategy θ^* such that

$$V(x,y) = E[U(X^{\theta*}_N)|X_0 = x,\, Y_0 = y]$$

Similarly, suppose that an investor holds a derivative with payoff $H(Y)$ and keeps it in his/her portfolio until maturity N, then the value of his/her investments at time 0 is given by

$$V^{\mathrm{d}}(x,y) = \sup_{\phi} \mathbb{E}[U(X^{\phi}_N + H(Y))|X^{\phi}_0 = x,\, Y_0 = y].$$

Also in this case we have shown that, if $H(Y) = \sum_{k=0}^{N-1} g(k, Y_k)$ with non-negative g, there exists an optimal strategy ϕ^* such that

$$V^{\mathrm{d}}(x,y) = \mathbb{E}[U(X_N^{\phi*} + H(Y))|X_0^{\phi*} = x,\, Y_0 = y].$$

In this case the utility indifference buying price $p^{\mathrm{b}}(x,y)$ is the solution to

$$V^{\mathrm{d}}(x - p^{\mathrm{b}}(x,y), y) = V(x,y).$$

That is, $p^{\mathrm{b}}(x,y)$ is the price at which an investor is indifferent between paying p^{b} now for receiving the claim from the derivative at expiration and not having the claim.

Definition 4.2. The utility indifference price of the claim with payoff $H(Y)$ is the price for which the two value functions V^{d} and V coincide. That is $V^{\mathrm{d}}(x - p^{\mathrm{b}}(x,y), y) = V(x,y)$ for all initial wealth levels x and initial precipitation y.

Proposition 4.4. The indifference buying and selling price of a contingent claim with a payoff $\sum_{n=0}^{N-1} g(n, Y_n)$ written on $(Y_k)_{k=0}^{N-1}$ is given by

$$p^{\mathrm{b}}(x,y) = \frac{1}{\alpha}\ln\frac{\mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{j=0}^{N-1}\frac{\mu^2(Y_j)}{\sigma^2(Y_j)}\right)|\, Y_0 = y\right]}{\mathbb{E}\left[\exp\left(-\sum_{j=0}^{N-1}\left(\frac{1}{2}\frac{\mu(Y_j)^2}{\sigma(Y_j)^2} + \alpha g(j,\, Y_j)\right)\right)|\, Y_0 = y\right]}$$

and

$$p^{\mathrm{s}}(x,y) = \frac{1}{\alpha}\ln\frac{\mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{j=0}^{N-1}\frac{\mu^2(Y_j)}{\sigma^2(Y_j)}\right)|\, Y_0 = y\right]}{\mathbb{E}\left[\exp\left(-\sum_{j=0}^{N-1}\left(\frac{1}{2}\frac{\mu(Y_j)^2}{\sigma(Y_j)^2} - \alpha g(j,\, Y_j)\right)\right)|\, Y_0 = y\right]},$$

respectively.

In particular, the prices do not depend on x.

Proof. This follows immediately from Propositions 4.1, 4.2 and 4.3. □

Suppose we rewrite

$$p^{\mathrm{b}}(x,y;\alpha) = -\frac{1}{\alpha}\ln\frac{\mathbb{E}\left[\exp\left(-\alpha\sum_{j=0}^{N-1} g(j,\, Y_j) - \frac{1}{2}\sum_{j=0}^{N-1}\frac{\mu(Y_j)^2}{\sigma(Y_j)^2}\right)|\, Y_0 = y\right]}{\mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{j=0}^{N-1}\frac{\mu^2(Y_j)}{\sigma^2(Y_j)}\right)|\, Y_0 = y\right]}. \tag{8}$$

Define a probability measure $\mathbb{Q}$ with density relative to $\mathbb{P}$ by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \frac{\exp\left(-\frac{1}{2}\sum_{j=0}^{N-1}\frac{\mu^2(Y_j)}{\sigma^2(Y_j)}\right)}{\mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{j=0}^{N-1}\frac{\mu^2(Y_j)}{\sigma^2(Y_j)}\right)\mid Y_0 = y\right]} \tag{9}$$

then we can rewrite Equation (8) as

$$p^{\mathrm{b}}(x,y;\alpha) = -\frac{1}{\alpha}\ln\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\alpha\sum_{j=0}^{N-1}g(j,\,Y_j)\mid Y_0 = y\right)\right].$$

Taking limits on both sides as $\alpha \to 0+$,

$$\begin{aligned}
\lim_{\alpha\to 0+} p^{\mathrm{b}}(x,y;\alpha) &= -\lim_{\alpha\to 0+}\frac{\ln\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\alpha\sum_{j=0}^{N-1}g(j,Y_j)\mid Y_0=y\right)\right]}{\alpha}\\
&= -\lim_{\alpha\to 0+}\frac{\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\alpha\sum_{j=0}^{N-1}g(j,Y_j)\mid Y_0=y\right)\right]}{\frac{\mathrm{d}}{\mathrm{d}\alpha}\alpha}\\
&= -\lim_{\alpha\to 0+}\frac{\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\alpha\sum_{j=0}^{N-1}g(j,Y_j)\mid Y_0=y\right)\right]}{\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\alpha\sum_{j=0}^{N-1}g(j,Y_j)\mid Y_0=y\right)\right]}\\
&= -\lim_{\alpha\to +0}\frac{\mathbb{E}_{\mathbb{Q}}\left[-\sum_{j=0}^{N-1}g(j,Y_j)\exp\left(-\alpha\sum_{j=0}^{N-1}g(j,Y_j)\mid Y_0=y\right)\right]}{\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\alpha\sum_{j=0}^{N-1}g(j,Y_j)\mid Y_0=y\right)\right]}\\
&= \mathbb{E}_{\mathbb{Q}}\left[\sum_{j=0}^{N-1}g(j,\,Y_j)\mid Y_0 = y\right].
\end{aligned}$$

We call

$$\lim_{\alpha\to 0+} p^{\mathrm{b}}(x,y) = \mathbb{E}_{\mathbb{Q}}\left[\sum_{j=0}^{N-1}g(j,\,Y_j)\mid Y_0 = y\right] \tag{10}$$

a zero risk-aversion limit price or $\mathbb{Q}$-risk-neutral price.

If $\mathbb{E}_{\mathbb{Q}}\left[\exp\left(\alpha\sum_{j=0}^{N-1}g(j,Y_j)\right)\mid Y_0 = y\right] < \infty$ for some $\alpha > 0$, as is the case if, for example, Assumption 4.1 holds, then by dominated convergence the limit for $\alpha \to 0+$ of the seller's price exists as well and is also equal to $\lim_{\alpha\to 0+} p^{\mathrm{b}}(x,y)$.

In the next section we discuss the numerical implementation of the above pricing formulas.

5. Some Numerical Illustrations

We are now ready to apply the techniques developed in this article to a (hypothetical) rainfall contract written on precipitation measured at Dagoretti weather station in Kenya. As our traded asset we use a share of a major Kenyan electricity company (KPLC), because Kenya produces most of its electricity from hydroelectric power plants. The contract pays the buyer 1 Ksh per millimeter of cumulative monthly rainfall above K mm for 12 months, that is

$$\sum_{j=0}^{N-1} g(j, Y_j) = \sum_{j=0}^{11} (Y_j - K)^+.$$

The computation using Monte Carlo simulation is rather straightforward: First we estimate all the parameters as described in Section 3, then we only need to compute the following expectations:

$$\mathbb{E}\left[\exp\left(-\sum_{j=0}^{N-1} \frac{1}{2}\frac{\mu^2(Y_j)}{\sigma^2(Y_j)}\right) | Y_0 = y\right], \quad \mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{j=0}^{N-1} \frac{\mu^2(Y_j)}{\sigma^2(Y_j)} \pm \alpha g(j, Y_j)\right) | Y_0 = y\right].$$

They can be computed in one Monte Carlo loop as each random precipitation path $(Y_0, \ldots, Y_{N-1})$ may be used three times over.

Note that Assumption 4.1 need not be fulfilled for every α. If for example $g(k, y) = y$, then the expectation $\mathbb{E}[U(\sum_{k=0}^{N-1} g(k, Y_k))]$ need not be finite because for a gamma-distributed random variable X with parameters a, b the expectation $\mathbb{E}[e^{-\alpha X}]$ is finite iff $\alpha < \frac{1}{b}$. We make the following, more precise statement:

Theorem 5.1 Assume the Markovian gamma model where the correlation structure is generated by Example 2.1. Let $|g(j, y_j)| \leq c + dy_j$ for all $j = 0, \ldots, N-1$.

Then, for $\alpha > 0$ small enough, we have

$$\mathbb{E}\left[\exp\left(\alpha \sum_{j=0}^{N-1} g(j, Y_j) | Y_0 = y\right)\right] < \infty.$$

For the proof we need the following lemma:

Lemma 5.2 Let $\alpha_1 \le \alpha_2$ and $\beta_1 \le \beta_2$. If F_j is the CDF of a gamma-distributed random variable with parameters α_j, β_j, $j = 1, 2$, then $F_1(x) \ge F_2(x)$ for all $x \ge 0$ (and $F_1^{-1}(y) \le F_2^{-1}(y)$ for all $y \in (0, 1)$).

Proof Suppose $\alpha_1 = \alpha_2$. Then the assertion is trivial. Next suppose $\beta_1 = \beta_2 = 1$, $\alpha_2 = \alpha_1 + \varepsilon$. We want to show that

$$\int_x^\infty \frac{1}{\Gamma(\alpha_1)} \xi^{\alpha_1 - 1} \mathrm{e}^{-\xi}\, \mathrm{d}\xi < \int_x^\infty \frac{1}{\Gamma(\alpha_1 + \varepsilon)} \eta^{\alpha_1 + \varepsilon - 1}\, \mathrm{e}^{-\eta}\, \mathrm{d}\eta$$

that is

$$\begin{aligned}
& \Gamma(\alpha_1 + \varepsilon) \int_x^\infty \xi^{\alpha_1 - 1} \mathrm{e}^{-\xi}\, \mathrm{d}\xi && < && \Gamma(\alpha_1) \int_x^\infty \eta^{\alpha_1 + \varepsilon - 1} \mathrm{e}^{-\eta}\, \mathrm{d}\eta \\
\Leftrightarrow\ & \int_0^\infty \eta^{\alpha_1 + \varepsilon - 1} \mathrm{e}^{-\eta}\, \mathrm{d}\eta \int_x^\infty \xi^{\alpha_1 - 1} e^{-\xi}\, \mathrm{d}\xi && < && \int_0^\infty \xi^{\alpha_1 - 1} \mathrm{e}^{-\xi}\, \mathrm{d}\xi \int_x^\infty \eta^{\alpha_1 + \varepsilon - 1} \mathrm{e}^{-\eta}\, \mathrm{d}\eta \\
\Leftrightarrow\ & \int_0^\infty \int_x^\infty (\xi\eta)^{\alpha_1 - 1} \mathrm{e}^{-(\xi + \eta)} \eta^{\varepsilon}\, \mathrm{d}\xi\, \mathrm{d}\eta && < && \int_0^\infty \int_x^\infty (\xi\eta)^{\alpha_1 - 1} \mathrm{e}^{-(\xi + \eta)} \eta^{\varepsilon}\, \mathrm{d}\eta\, \mathrm{d}\xi \\
\Leftrightarrow\ & \int_0^x \int_x^\infty (\xi\eta)^{\alpha_1 - 1} \mathrm{e}^{-(\xi + \eta)} \eta^{\varepsilon}\, \mathrm{d}\xi\, \mathrm{d}\eta && < && \int_0^x \int_x^\infty (\xi\eta)^{\alpha_1 - 1} \mathrm{e}^{-(\xi + \eta)} \eta^{\varepsilon}\, \mathrm{d}\eta\, \mathrm{d}\xi.
\end{aligned}$$

But the last inequality is certainly true, as

$$\begin{aligned}
\int_0^x \int_x^\infty (\xi\eta)^{\alpha_1 - 1} \mathrm{e}^{-(\xi + \eta)} \eta^{\varepsilon}\, \mathrm{d}\xi\, \mathrm{d}\eta &< \int_0^x \int_x^\infty (\xi\eta)^{\alpha_1 - 1} \mathrm{e}^{-(\xi + \eta)} x^{\varepsilon}\, \mathrm{d}\xi\, \mathrm{d}\eta \\
&< \int_0^x \int_x^x (\xi\eta)^{\alpha_1 - 1} \mathrm{e}^{-(\xi + \eta)} \eta^{\varepsilon}\, \mathrm{d}\eta\, \mathrm{d}\xi.
\end{aligned}$$

The remainder of the proof is trivial: $F_{\alpha_1, \beta_1} \ge F_{\alpha_1, \beta_2} \ge F_{\alpha_2, \beta_2}$. □

Proof of Theorem 5.1 Obviously,

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\alpha \sum_{j=0}^{N-1} g(j, Y_j)\right) | Y_0 = y\right] &\le \mathbb{E}\left[\exp\left(\alpha \sum_{j=0}^{N-1} c + dY_j\right) | Y_0 = y\right] \\
&= \mathrm{e}^{\alpha N c} \mathbb{E}\left[\exp\left(\alpha d \sum_{j=0}^{N-1} Y_j\right) | Y_0 = y\right]
\end{aligned}$$

and we want to show that the last term is finite for small α.

First we show that $\mathbb{E}\left[\exp\left(\alpha d \sum_{j=0}^{N-1} Y_j\right)\right] < \infty$ for α small enough.

Let G be the CDF of a gamma-distributed random variable with parameters $\hat{\alpha} := \max_{0\leq j\leq N-1} \alpha_j$ $\hat{\beta} := \max_{0\leq j\leq N-1} \beta_j$, where α_j, β_j are the parameters of F_j. Then $G^{-1}(y) \geq F_j^{-1}(y)$ and therefore $Z_j := G^{-1}(F_j(Y_j)) \geq F_j^{-1}(F_j(Y_j)) = Y_j$, such that

$$\mathbb{E}\left[\exp\left(\alpha d \sum_{j=0}^{N-1} Y_j\right)\right] \leq \mathbb{E}\left[\exp\left(\alpha d \sum_{j=0}^{N-1} Z_j\right)\right],$$

and $Z_0, \ldots, Z_{N-1}$ are gamma-distributed random variables with parameters $\hat{\alpha}$, $\hat{\beta}$. Let $M := \max_{0\leq j\leq N-1} Z_j$ and denote by f_M, F_M the probability distribution function and the CDF of M, respectively.

Then $1 - F_M(y) = \mathbb{P}(M > y) \leq N\mathbb{P}(Z_0 > y) = N(1 - G(y))$. Therefore

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\alpha d \sum_{j=0}^{N-1} Z_j\right)\right] &\leq \mathbb{E}\left[e^{\alpha d N M}\right] \\
&= \int_0^\infty e^{\alpha d N y} f_M(y) dy \\
&= e^{\alpha d N y}(1 - F_M(y))|_0^\infty + \int_0^\infty \alpha d N\, e^{\alpha d N y}(1 - F_M(y)) dy \\
&\leq N e^{\alpha d N y}(1 - G(y))|_0^\infty + \int_0^\infty \alpha d N^2\, e^{\alpha d N y}(1 - G(y)) dy,
\end{aligned}$$

which is finite for α small enough.

We have established $\mathbb{E}\left[\exp\left(\alpha d \sum_{j=0}^{N-1} Y_j\right)\right] < \infty$. Now, as

$$\infty > \mathbb{E}\left[\exp\left(\alpha d \sum_{j=0}^{N-1} Y_j\right)\right] = \int_0^\infty \mathbb{E}\left[\exp\left(\alpha \mathrm{d} \sum_{\mathrm{j}=0}^{\mathrm{N}-1} \mathrm{Y_j}\right) \mid Y_0 = y\right] f_{\gamma_0,\beta_0}(y) dy$$

we need to have $\mathbb{E}\left[\exp\left(\alpha d \sum_{j=0}^{N-1} Y_j\right) | Y_0 = y\right] < \infty$ for almost all y. But as $y \mapsto \mathbb{E}\left[\exp\left(\alpha d \sum_{j=0}^{N-1} Y_j\right) | Y_0 = y\right]$ is strictly increasing, this means that $\mathbb{E}\left[\exp\left(\alpha d \sum_{j=0}^{N-1} Y_j\right) | Y_0 = y\right] < \infty$ for all y. (Here we use Remark 2.1, i.e. a property of the Gaussian copula).

5.1 Accelerating the Monte Carlo Simulation for Pricing Rainfall Options

The Monte Carlo simulation for our model can be rather time-consuming. We provide some hints for speeding up the algorithm.

First, we recall that to generate a random precipitation path we have to evaluate $N-1$ inverses of the CDFs of gamma-distributed random variables. The use of interpolating functions instead of the original CDFs usually yields a great improvement in efficiency.

Second, there are near-at-hand control variates for our expectations: For example use

$$\mathbb{E}\left[\exp\left(-\sum_{j=0}^{N-1}\frac{1}{2}\frac{\mu^2(Z_j)}{\sigma^2(Z_j)}\right)|Z_0=y\right],$$

where $Z_1, Z_2, \ldots, Z_{N-1}$ are *independent* random variables having the same respective distributions as $Y_1, \ldots, Y_{N-1}$. As the modulus of the correlation of the Y_j's is typically smaller than 0.1, this control is very strongly correlated to the original variable.

5.2 Numerical Results

In our examples, the method in Section 5.1 has made the variance small enough to make a couple of thousand runs per price calculation sufficient to guarantee an error of less than 1%.

The effect of the variance reduction by the control variates is diminished if ρ becomes too big. For example for $\rho = 0$ we can compute the price exactly and need no simulation at all and for $\rho = 0.4$ we already need around 10^5 to have an error of less than 1%. The historical rainfall data from Dagoretti weather station gave an estimate for ρ of about 0.06%.

Table 1 shows the buyer's and seller's prices p^{b} and p^{s}, respectively, in our setup for $\alpha = 0.001$ and for several values of ρ. The option has payoff $\sum_{k=0}^{11}(Y_k - K)_+$ with $K = 100$ and $Y_0 = 100$. For comparison we also listed the buyer's and seller's prices without hedging, that is

$$q^{\mathrm{b}}(y) = -\frac{1}{\alpha}\log\mathbb{E}\left[\exp\left(-\alpha\sum_{j=0}^{N-1}g(k, Y_k)\right)|\,Y_0=y\right],$$

$$q^{\mathrm{s}}(y) = -\frac{1}{\alpha}\log\mathbb{E}\left[\exp\left(\alpha\sum_{j=0}^{N-1}g(k, Y_k)\right)|\,Y_0=y\right].$$

Table 1. Dependence of prices on correlation.

ρ	p^{b}	p^{s}	q^{b}	q^{s}
0.0	340.1	390.8	344.7	396.1
0.01	340.2	391.3	344.8	396.7
0.05	340.5	393.6	345.5	399.5
0.1	341.0	396.5	346.4	403.0
0.2	341.5	402.2	347.8	409.9
0.3	342.0	408.2	349.4	417.5
0.4	341.6	413.0	350.3	424.2

Table 2. The Dagoretti rainfall option's pricing results, $\rho = 0.06$.

		With seasonality						Without seasonality					
		With hedge		$\mathbb{Q}$-risk neutral	Without hedge		$\mathbb{P}$-risk neutral	With hedge		$\mathbb{Q}$-risk neutral	Without hedge		$\mathbb{P}$-risk neutral
K	α	p^s	p^b		q^s	q^b		p^s	p^b		q^s	q^b	
50	0.001	652.268	584.353	615.799	660.068	591.074	623.001	655.197	557.254	614.845	669.955	569.398	601.521
	0.0005	633.307	599.536	615.799	640.789	606.485	623.001	627.004	578.375	614.845	641.009	591.081	601.521
	0.0001	619.194	612.454	615.799	626.451	619.602	623.001	606.42	596.716	614.845	619.874	609.911	601.521
100	0.001	394.246	340.476	365.226	400.247	345.606	370.691	377.731	309.666	349.787	388.497	318.313	340.193
	0.0005	379.108	352.429	365.226	384.825	357.668	370.691	357.945	324.181	349.787	368.093	333.277	340.193
	0.0001	367.909	362.586	365.226	373.423	368.002	370.691	343.596	336.86	349.787	353.295	346.35	340.193
150	0.001	256.998	217.302	235.391	261.499	220.992	239.441	238.079	192.611	219.598	245.661	198.61	212.897
	0.0005	245.686	225.968	235.391	249.947	229.829	239.441	224.774	202.234	219.598	231.889	208.565	212.897
	0.0001	237.377	233.44	235.391	241.467	237.45	239.441	215.168	210.673	219.598	221.949	217.297	212.897
200	0.001	165.153	137.507	150.021	168.409	140.139	152.928	151.384	121.649	139.479	156.609	125.748	134.882
	0.0005	157.207	143.475	150.021	160.279	146.235	152.928	142.655	127.92	139.479	147.547	132.254	134.882
	0.0001	151.405	148.664	150.021	154.344	151.539	152.928	136.368	133.429	139.479	141.021	137.971	134.882
250	0.001	104.776	86.2802	94.6091	107.072	88.1113	96.646	96.3852	77.199	88.8702	99.9483	79.9938	85.7351
	0.0005	99.4249	90.2462	94.6091	101.584	92.1743	96.646	90.7504	81.2436	88.8702	94.0867	84.1993	85.7351
	0.0001	95.542	93.6947	94.6091	97.6037	95.7072	96.646	86.6937	84.7975	88.8702	89.8669	87.8951	85.7351
300	0.001	65.8811	53.8379	59.2432	67.4602	55.0893	60.6394	61.4834	49.1203	56.7394	63.8938	51.015	54.6164
	0.0005	62.3824	56.4095	59.2432	63.8647	57.7292	60.6394	57.8486	51.7238	56.7394	60.1065	53.7265	54.6164
	0.0001	59.8445	58.6541	59.2432	61.2582	60.0332	60.6394	55.2342	54.0122	56.7394	57.3826	56.1104	54.6164

We see that the buyer's prices are relatively insensitive to the correlation although the seller's prices admit a sizable dependence on the correlation.

The results as given in Table 2 show a relative decrease in the price because of hedging with an electricity contract. Both seller's and buyers price with the power hedge is lower than without it. The gap between buyer's and seller's price for each strike price decreases but remains positive. In addition, the price table shows a relative difference in buyer's price for the model with seasonality as compared to the model without seasonality. The difference increases with risk aversion. However, for low risk aversion level, some seller price with hedge is lower than the buyer price without hedging. Hence, an acceptance weather derivatives deal can be done between a seller with knowledge and access to the power market and a buyer without the access or knowledge, provided that the seller hedges himself/herself in the power market.

In addition, we note that for some risk aversion the seller's price does not exist (i.e. if, for example $g(k,y)=y$, then the expectation $\mathbb{E}\left[U\left(\sum_{k=0}^{N-1} g(k,Y_k)\right)\right]$ need not be finite because for a gamma-distributed random variable X with parameters a, b the expectation $\mathbb{E}[\mathrm{e}^{\alpha X}]$ is finite iff $\alpha < \frac{1}{b}$).

We observe from Table 2 that the difference between hedged and unhedged prices is still noticeable but admittedly not very big. The difference would probably have been bigger if we could have correlated to electricity prices directly instead of the stock price of an electric power producer.

The risk premia for the buyer/seller can also be read off from Table 2. By this we mean the difference between the utility indifference price and the $\mathbb{Q}$- risk-neutral price, $\mathbb{E}_{\mathbb{Q}}\left[\sum_{k=0}^{N-1} g(k,Y_k) | Y_0 = y\right]$, where the change of measure is given by Equation (9).

6. Discussion

We have developed a new discrete-time model for precipitation and derived a pricing formula for a class of rainfall options using utility indifference pricing for an investor with exponential utility. We implemented an efficient version of Monte Carlo simulation for the rainfall process to compute expectations for the pricing formula in Section 4.

This provides an alternative to the model by Carmona and Diko (2005). Our model applies to low-frequency/long-term data as compared to Carmona–Diko model that applies to high-frequency/short-term data. Ideally, a model should take both information about long-term behaviour and short-term behaviour into account. But we focused more on the effect of seasonality of the rainfall process: Whereas our model is in a sense more coarse than the Carmona–Diko model, it has the advantage that it allows for seasonal variation, which is obviously of paramount importance in some parts of the world.

We therefore view both models as important steps in the development of more refined rainfall process models, for example a Markov jump process model like the Carmona–Diko model, which allows for seasonal variation, or a discrete model like in the present study that allows for efficient calibration of long-term/high-frequency data.

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Note

[1]In our numerical examples we used a censoring level of $A = 0.1$. That means that we assume that a precipitation of less than 0.1 mm per month cannot be detected by the instruments.

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