Hindawi
Journal of Applied Mathematics
Volume 2019, Article ID 8246578, 17 pages
https://doi.org/10.1155/2019/8246578

Research Article

A Multicurve Cross-Currency LIBOR Market Model

Charity Wamwea [1] **Philip Ngare,** [2] **and Martin Le Doux Mbele Bidima** [3]

[1] *Department of Statistics and Actuarial Sciences, Jomo Kenyatta University of Agriculture and Technology, Kenya*
[2] *School of Mathematics, The University of Nairobi, Kenya*
[3] *Department of Mathematics, University of Yaoundé I, Cameroon*

Correspondence should be addressed to Charity Wamwea; wamweac@gmail.com

Received 23 October 2018; Revised 29 January 2019; Accepted 11 February 2019; Published 10 March 2019

Academic Editor: Qingbiao Wu

After the dawn of the August 2007 financial crisis, banks became more aware of financial risk leading to the appearance of nonnegligible spreads between LIBOR and OIS rates and also between LIBOR of different tenors. This consequently led to the birth of multicurve models. This study establishes a new model; the multicurve cross-currency LIBOR market model (MCCCLMM). The model extends the initial LIBOR Market Model (LMM) from the single-curve cross-currency economy into the multicurve cross-currency economy. The model incorporates both the risk-free OIS rates and the risky forward LIBOR rates of two different currencies. The established model is suitable for pricing different quanto interest rate derivatives. A brief illustration is given on the application of the MCCCLMM on pricing quanto caplets and quanto floorlets using a Black-like formula derived from the MCCCLMM.

1. Introduction

Modeling of LIBOR rates has evolved since their inception in the late nineties. However, the most notable change occurred in August 2007 where a severe financial crisis caused a number of anomalies in the interest rate markets. Before the 2007 credit crunch, the spreads between the LIBOR rates and the overnight indexed swap (OIS) rates were negligible. In addition to this, the spreads between different LIBOR curves of different tenors were also considered to be negligible. Hence, a single interest rate curve was sufficient for both discounting and generating future cash flows.

However, after the 2007 financial crisis, the LIBOR-OIS spreads of different maturities began to evolve randomly over time. In addition to this, the spreads between different LIBOR curves of different tenors took the same fate. These spreads became substantially too large to be ignored making the negligibility assumption no longer hold. See images by [1] in Figures 1 and 2.

Hence, the possibility of using one curve for both discounting and generating future cash flows was greatly challenged. This led to the introduction of the multiple curve interest rate models. In this new framework, one curve is used for discounting (mostly the OIS (risk-free) curve) and the other curve(s) used for generating future cash flows (mostly different LIBOR curves of different tenors).

So far, there are various models proposed in different pieces of literature with regard to modeling LIBOR under the multiple curve framework: see [2–12] and so on. However, to our knowledge, author of [13] was the first author to suggest this approach and this was ironically before the financial crisis occurred. It can be seen that all the proposed multicurve models can be categorized to fall either under the short rate approach such as [14–16] or the Heath-Jarrow Morton (HJM) approach proposed in the early nineties by [17] or the LIBOR market model (LMM) approach first proposed by [18, 19] in the late nineties.

In addition to this, it was also noted that, under the multiple curve framework, one may choose to model the OIS and LIBOR rates directly. This is said to lead to tractable pricing formulas. However, one can not guarantee the positivity of the LIBOR-OIS spreads. Alternatively, one may choose to model the OIS and LIBOR-OIS spreads directly and then later infer the dynamics of the LIBOR. This ensures that the positivity of the LIBOR-OIS spreads is maintained. However, the pricing formulas derived under this approach have been

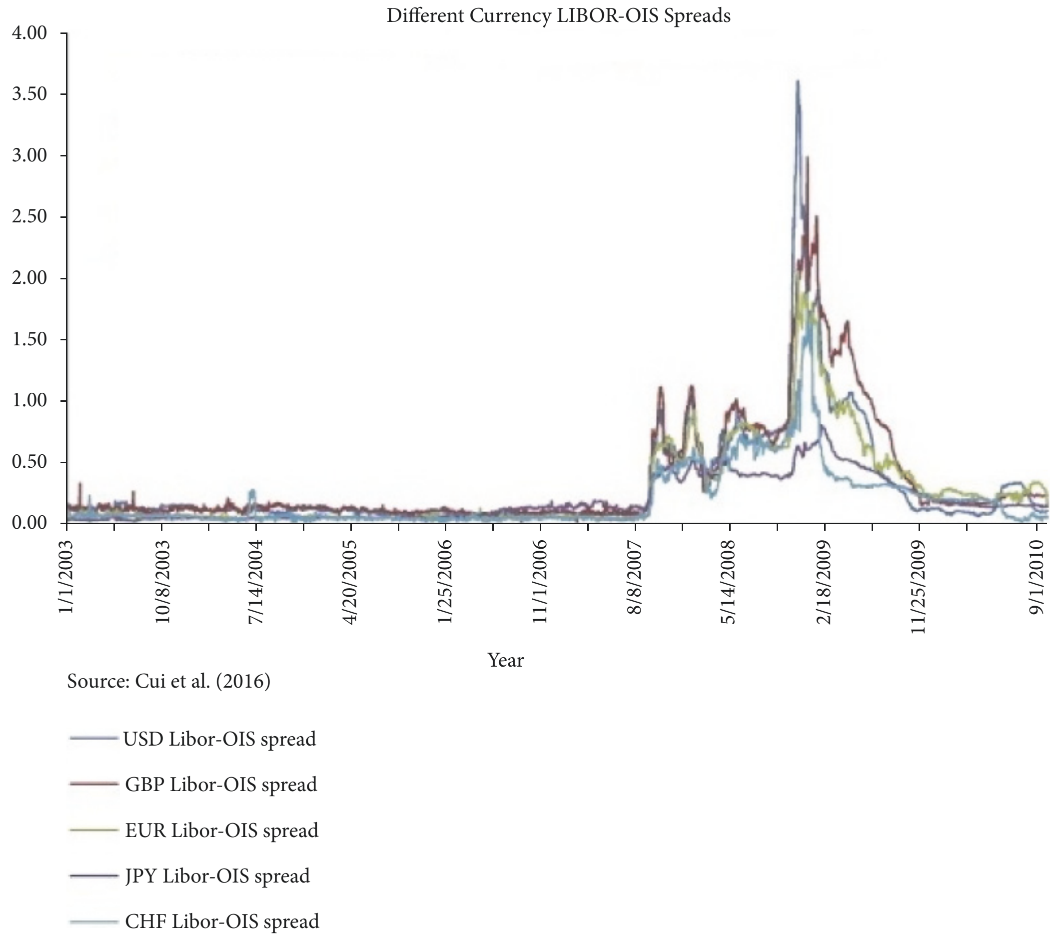

Figure 1: Different Currency LIBOR-OIS Spreads.

found to be less tractable. Finally, one may choose to model the LIBOR and LIBOR-OIS spreads directly and later infer the dynamics of the OIS rates. However, the positivity of the OIS rates cannot be guaranteed. In this paper, we follow the first approach.

The aim of this paper is to construct a multicurve cross-currency LIBOR market model under the spot domestic risk neutral measure. The model involves taking into account both the risk-free and the risky interest rates. For this purpose, we adapt the lognormal volatility model for the LIBOR and OIS rates. We also adapt the geometric Brownian motion model for the spot foreign exchange rate. We practically aim to extend the works by [20, 21] from the single-curve cross-currency framework into the multicurve cross-currency framework so that it conforms to the modern practices.

2. Problem Formulation

In this paper, our approach closely resembles that of the Heath-Jarrow-Morton and the LIBOR market model (HJM-LMM) approach in coming up with the multicurve cross-currency LIBOR market model (MCCCLMM).

2.1. Model Notations. Let $X(t)$ be the spot foreign exchange rate at time t quoted as the ratio of units of domestic currency to one unit of foreign currency. d and f denote the domestic and foreign markets, respectively. D and L denote the risk-free and risky curves, respectively. r_d^D and r_f^D are the domestic and foreign risk-free short rates of interest. r_d^L and r_f^L are the domestic and foreign risky short rates of interest. $B_d^D(t)$ and B_f^D are the corresponding risk-free domestic and foreign money market accounts. $s_d(t)$ and $s_f(t)$ are the spread between the domestic and foreign risky and risk-free short rates of interest, respectively. $B_d^L(t)$ and $B_f^L(t)$ are the corresponding risky domestic and foreign money market accounts. $P_d^D(t,T)$ and $P_f^D(t,T)$ denote the risk-free domestic and foreign zero-coupon bonds. $P_d^L(t,T)$ and $P_f^L(t,T)$ denote the risky domestic and foreign zero-coupon bonds. $L_d^D(t,T)$ and $L_f^D(t,T)$ are the domestic and foreign simply compounded t-forward rates associated with the domestic and foreign risk-free discount curves. Finally, $L_d(t,T)$ and $L_f(t,T)$ are the domestic and foreign simply compounded t-forward rates associated with the domestic and foreign risky fictitious zero-coupon bonds.

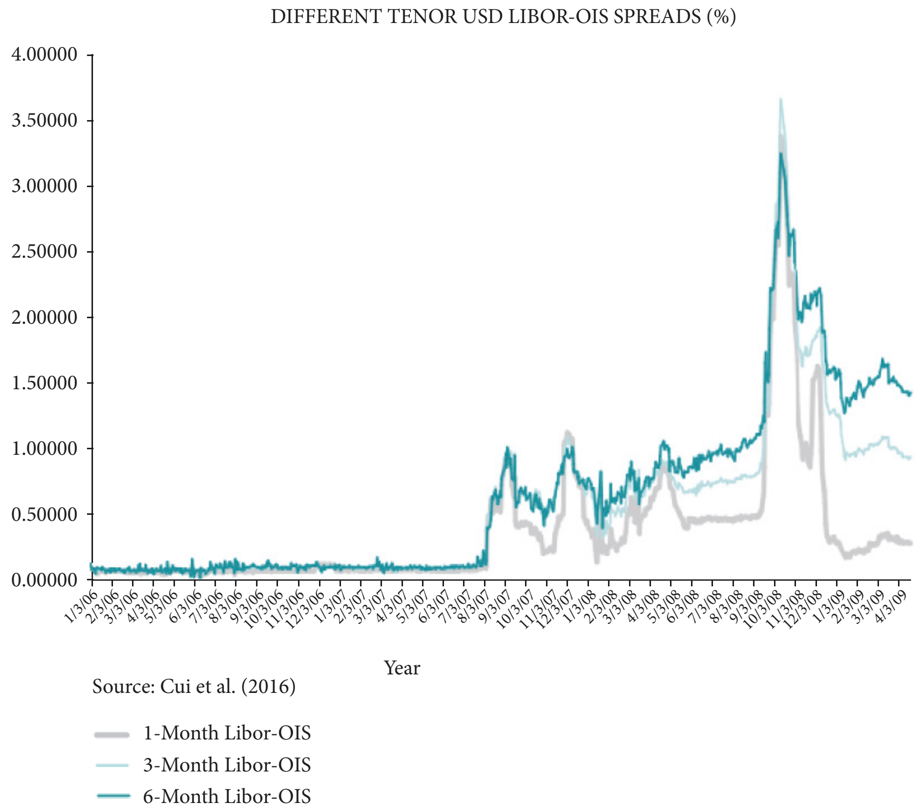

Figure 2: Different Tenor USD LIBOR-OIS Spreads.

2.2. Model Assumptions. We model a frictionless market, free of arbitrage opportunities. In this market, we assume that trading takes place continuously for a given time interval $[0, \mathcal{T}]$, where $\mathcal{T}$ is some positive final date. The market uncertainty is assumed to be modeled by the filtered probability space, $(\Omega, \mathcal{F}, (\mathcal{F}_{t\in[0,\mathcal{T}}, \mathbb{Q}_d)$. In our model, we consider only one tenor, τ. We also assume that there exist two markets, domestic (d) and foreign (f), which can be linked with the foreign exchange rate markets. In addition to this, we assume that there exist both risk-free and risky rates in the two markets. The OIS rate is taken to be the most preferable risk-free rate and it is the one that is used to construct the discount curve. On the other hand, we take the LIBOR of a single tenor to be the risky rate and this is the rate that is used to generate the future cash flows.

We assume that there exists, at any time t, a risk-free (domestic or foreign) zero-coupon bond $P_k^D(\cdot, T) \geq 0$; $k \in \{d, f\}$ such that $P_k^D(T, T) = 1$ for all $T \in [0, \mathcal{T}]$. Assuming that the mapping $T \longmapsto P_k^D(t, T)$ at any time $t \in [0, \mathcal{T}]$ is differentiable, then the simply compounded risk-free forward interest rate is given as

$$L_k^D(t, T) = \frac{1}{\tau}\left(\frac{P_k^D(t, T)}{P_k^D(t, T+\tau)} - 1\right); \quad k \in \{d, f\} \qquad (1)$$

where the risk-free domestic or foreign zero-coupon bond price in [6] is defined as

$$P_k^D(t, T) = \mathbb{E}\left[e^{-\int_t^T r_k^D(s)ds} \mid \mathcal{F}_t\right]; \quad k \in \{d, f\} \qquad (2)$$

We further assume that there exists a risk-free (domestic or foreign) money market account $B_k^D(t)$ whose price process is given by

$$B_k^D(t) = e^{\int_0^t r_k^D(s)ds}; \quad k \in \{d, f\} \qquad (3)$$

where $r_k^D(t)$ is the (domestic or foreign) risk-free short rate of interest.

We assume that there also exists, at any time t, a risky fictitious (domestic or foreign) zero-coupon bond $P_k^L(\cdot, T) \geq 0$; $k \in \{d, f\}$. Assuming that the mapping $T \longmapsto P_k^L(t, T)$ at any time $t \in [0, \mathcal{T}]$ is differentiable, then the fictitious domestic or foreign bond in [22] is defined as

$$P_k^L(t, T) = \mathbb{E}\left[e^{-\int_t^T (r_k^D(u)+s_k(u))du} \mid \mathcal{F}_t\right]; \quad k \in \{d, f\} \qquad (4)$$

We further assume that there exists a risky (domestic or foreign) money market account $B_k^L(t)$ whose price process is given by

$$B_k^L(t) = e^{\int_0^t r_k^L(s)ds}, \quad k \in \{d, f\} \qquad (5)$$

where $r_k^L(t) = r_k^D(t) + s_k(t)$ is the (domestic or foreign) risky short rate of interest and $s_k(t)$ is the k LIBOR-OIS short rate spread.

Finally, given the filtered probability space $(\Omega, \mathcal{F}_t, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, it is assumed that the dynamics of the entire economy under the objective measure, $\mathbb{P}$, associated with the real world probabilities is given by

$$\begin{aligned} dB_k^j(t) &= r_k^j(t) B_k^j(t); \quad k \in \{d, f\} \;\; \& \;\; j \in \{D, L\} \\ dP_k^j(t,T) &= \mu_{kj}(t,T) P_k^j(t,T)\, dt \\ &\quad - \sigma_{kj}(t,T) P_k^j(t,T)\, d\overline{W}_{kj}(t) \\ dX(t) &= \mu_x(t) X(t)\, dt + \sigma_x(t) X(t)\, d\overline{W}_{xx}(t) \\ dL_k^D(t,T) &= \theta_{kD}(t,T)\, dt + \gamma_{kD}(t,T)\, d\overline{W}_{kD}(t) \\ dL_k(t,T) &= \theta_{kL}(t,T)\, dt + \gamma_{kL}(t,T)\, d\overline{W}_{kL}(t) \end{aligned} \tag{6}$$

2.3. Conditions for No Arbitrage. We see that we have five sources of risk: the domestic risk-free markets, domestic risky markets, foreign risk-free markets, foreign risky markets, and the foreign exchange rate markets and according to Meta Theorem [23], if A is the number of underlying traded assets excluding the risk-free asset and R is the number of random sources of risk, then we have the following.

(i) The model is free of arbitrage-free if and only if $A \leq R$.

(ii) The model is complete if and only if $A \geq R$.

(iii) The model is complete and arbitrage-free if and only if $A = R$.

Hence assuming that all these markets are correlated, then by Meta's theorem, a 5-dimensional correlated Wiener process must be used to ensure that the model is complete and arbitrage-free. The correlated 5-dimensional Wiener process W is defined as

$$W(t) = \begin{bmatrix} W_{dD}(t) \\ W_{dL}(t) \\ W_{fD}(t) \\ W_{fL}(t) \\ W_{xx}(t) \end{bmatrix} \tag{7}$$

such that

$$d[W_i, W_j]_t = \begin{cases} \rho_i^j dt = \rho_j^i dt; & \text{for } i \neq j \;\&\; i, j \in \{dD, fD, dL, fL, xx\} \\ dt; & i = j \end{cases} \tag{8}$$

Also, according to the First Fundamental Theorem [23], a model is arbitrage-free if and only if there exists an equivalent (local) martingale measure $\mathbb{Q}$. Hence, assuming that the risk-free assets in both the foreign and domestic markets are basic traded instruments and according to [4, 6], the risky LIBOR bonds in the two markets are fictitious (i.e., nontraded) assets. Assuming further that there exists a usual domestic risk neutral probability measure $\mathbb{Q}_d^D \sim \mathbb{P}$, then to ensure that there is no arbitrage, we have the following.

(i) Under $\mathbb{Q}_d^D$, all domestically traded assets with a price process of say $\Pi(t)$ must have the domestic risk-free short rate of interest, $r_d^D(t)$, as its local rate of return [23]. That is, its $\mathbb{Q}_d^D$ dynamics will be of the form

$$d\Pi(t) = r_d^D(t)\Pi(t)\, dt + \sigma_\Pi(t)\Pi(t)\, dW(t) \tag{9}$$

where the volatility vector σ_Π is the same as the one under the objective measure $\mathbb{P}$.

(ii) Under $\mathbb{Q}_d^D$, all the domestic fictitious assets with a price process of say $\Pi(t)$ must have the domestic risky short rate of interest, $r_d^L(t)$, as the local rate of interest. That is, its $\mathbb{Q}_d^D$ dynamics will be of the form

$$d\Pi(t) = r_d^L(t)\Pi(t)\, dt + \sigma_\Pi(t)\Pi(t)\, dW(t) \tag{10}$$

where the volatility vector σ_Π is the same as the one under the objective measure $\mathbb{P}$.

(iii) Under $\mathbb{Q}_d^D$, all normalized asset price processes of all domestically traded assets with a price process of say $\Pi(t)$ discounted using the domestic risk-free money market account, $B_d^D(t)$, as the numeraire, must be $\mathbb{Q}_d^D$ martingales [21, 23]. That is, the normalized price process

$$Z_\Pi(t) = \frac{\Pi(t)}{B_d^D(t)} \tag{11}$$

is a $\mathbb{Q}_d^D$ martingale.

(iv) Under $\mathbb{Q}_d^D$, all normalized asset price processes of all domestic fictitious assets with a price process of say $\Pi(t)$ discounted using the domestic risky money market account, $B_d^L(t)$, as the numeraire, must be $\mathbb{Q}_d^D$ martingales [6]. That is, the normalized price process

$$Z_\Pi(t) = \frac{\Pi(t)}{B_d^L(t)} \tag{12}$$

is a $\mathbb{Q}_d^D$ martingale.

(v) All normalized asset price processes of all domestically traded assets discounted using the domestic risk-free zero-coupon bond, $P_d^D(t,T)$, are $\mathbb{Q}_d^{D^T}$ martingales. That is, the normalized price process

$$Z_\Pi(T) = \frac{\Pi(t)}{P_d^D(t,T)} \tag{13}$$

2.4. The Multicurve Cross-Currency LIBOR Market Model. In this section, a brief description of how the dynamics of the MCCCLMM is derived under the domestic risk neutral measure, $\mathbb{Q}_d^D$, satisfying the no arbitrage assumptions described in Section 2.3 above, is given. It is clear from our model assumptions that $B_d^D(t)$ and $P_d^D(t,T)$ are domestically traded assets. In addition to this, $B_d^L(t)$ and $P_d^L(t,T)$ are also assumed to be domestic fictitious assets. Hence from the no arbitrage conditions described in Section 2.3 requirements

(i) and (ii) and from the dynamics of the entire economy described in (6), we get the dynamics of the domestic assets under the spot domestic risk neutral measures to be given by

$$\begin{aligned} dB_d^D(t) &= r_d^D(t)\, B_d^D(t)\, dt \\ dB_d^L(t) &= r_d^L(t)\, B_d^L(t)\, dt \\ dP_d^D(t,T) &= r_d^D(t)\, P_d^D(t,T)\, dt \\ &\quad - \sigma_{dD}(t,T)\, P_d^D(t,T)\, dW_{dD}(t) \\ dP_d^L(t,T) &= r_d^L(t)\, P_d^L(t,T)\, dt \\ &\quad - \sigma_{dL}(t,T)\, P_d^L(t,T)\, dW_{dL}(t) \end{aligned} \tag{14}$$

Lemma 1. *Assuming that the market is arbitrage-free, then the possibility of investing in a certain foreign asset at a foreign risk-free short rate of interest should be equivalent to investing in a domestic asset with a price process $B_f^{*D}(t)$ (see [23]) where*

$$B_f^{*D}(t) = B_f^D(t)\, X(t) \tag{15}$$

Therefore, from the dynamics of the entire economy described in (6), it is trivial that the dynamics of the price process $B_f^{*D}(t)$ is given by

$$\begin{aligned} dB_f^{*D}(t) &= \left(\mu_x(t) + r_f^D(t)\right) B_f^{*D}(t)\, dt \\ &\quad + \sigma_x(t)\, B_f^{*D}(t)\, d\overline{W}_{xx}(t) \end{aligned} \tag{16}$$

However, since $B_f^{*D}(t)$ is also a domestically traded asset, then it should also satisfy the no arbitrage condition under Section 2.3 (i). Hence, under the usual domestic risk neutral measure, $\mathbb{Q}_d^D$,

$$dB_f^{*D}(t) = r_d^D(t)\, B_f^{*D}(t)\, dt + \sigma_x(t)\, B_f^{*D}(t)\, dW_{xx}(t) \tag{17}$$

and from (14), it is clear that the spot foreign exchange rate can be expressed as

$$X(t) = \frac{B_f^{*D}(t)}{B_f^D(t)} \tag{18}$$

and hence using the dynamics expressed in (13) and (16), we see that the dynamics of the spot foreign exchange rate under the domestic risk neutral measure, $\mathbb{Q}_d^D$, will be given by

$$\begin{aligned} dX(t) &= \left(r_d^D(t) - r_f^D(t)\right) X(t)\, dt \\ &\quad + \sigma_x(t)\, X(t)\, dW_{xx}(t) \end{aligned} \tag{19}$$

Lemma 2. *Assuming that the market is free of arbitrage, then the possibility of investing in a foreign risk-free zero-coupon bond should be equivalent to investing in a general risk-free domestic zero-coupon bond with a price process $P_f^{*D}(t,T)$ (see [23]) where*

$$P_f^{*D}(t,T) = P_f^D(t,T)\, X(t) \tag{20}$$

Therefore, using the dynamics of the entire economy described in (6), the dynamics of $P_f^{*D}(t,T)$ is derived to be

$$\begin{aligned} dP_f^{*D}(t,T) &= \left(\mu_{fD}(t,T) + \mu_x(t)\right) P_f^{*D}(t,T)\, dt \\ &\quad - \sigma_{fD}(t,T)\, P_f^{*D}(t,T)\, d\overline{W}_{fD}(t) \\ &\quad + \sigma_x(t)\, P_f^{*D}(t,T)\, d\overline{W}_{xx}(t) \end{aligned} \tag{21}$$

It should also be noted that $P_f^{*D}(t,T)$ is a domestically traded asset; hence, it should satisfy the no arbitrage condition discussed under Section 2.3 (i). Therefore, under the domestic risk neutral measure, $\mathbb{Q}_d^D$,

$$\begin{aligned} dP_f^{*D}(t,T) &= r_d^D(t)\, P_f^{*D}(t,T)\, dt \\ &\quad + \sigma_x(t)\, P_f^{*D}(t,T)\, dW_{xx}(t) \\ &\quad - \sigma_{fD}(t,T)\, P_f^{*D}(t,T)\, dW_{fD}(t) \end{aligned} \tag{22}$$

From (19), it is clear that the foreign risk-free zero-coupon bond can be expressed as

$$P_f^D(t,T) = \frac{P_f^{*D}(t,T)}{X(t)} \tag{23}$$

and hence using the dynamics expressed in (18) and (21), we find the dynamics of the foreign risk-free zero-coupon bond to be

$$\begin{aligned} &dP_f^D(t,T) \\ &\quad = \left(r_f^D(t) + \rho_{xx}^{fD}(t)\, \sigma_x(t)\, \sigma_{fD}(t,T)\right) P_f^D(t,T)\, dt \\ &\quad - \sigma_{fD}(t,T)\, P_f^D(t,T)\, dW_{fD}(t) \end{aligned} \tag{24}$$

Remark 3. In practice, we notice that $\tau L_d^D(t,T) P_d^D(t,T+\tau)$ and $\tau L_f^D(t,T) P_f^D(t,T+\tau) X(t)$ are the time t domestic prices of the floating leg part of the FRA rates associated with the risk-free simply compounded domestic and foreign forward rates: $L_d^D(t,T)$ and $L_f^D(t,T)$, respectively, where τ is the tenor. In addition to this, $\tau L_d(t,T) P_d^D(t,T+\tau)$ and $\tau L_f(t,T) P_f^D(t,T+\tau) X(t)$ are the time t domestic prices of the floating leg part of the FRA rates associated with the risky simply compounded domestic and foreign LIBOR forward rates: $L_d(t,T)$ and $L_f(t,T)$, respectively. Hence, as we can see, they are also domestically traded assets. Therefore, they should satisfy the no arbitrage condition defined under Section 2.3 (i).

Hence as a consequence of the above remark and in using Itô's Lemma, the dynamics of the different simply compounded forward rates of interest under the usual domestic risk neutral measure are found to be given by

$$\begin{aligned} dL_d^D(t,T) &= \sigma_{dD}(t,T+\tau)\gamma_{dD}(t,T)L_d^D(t,T)\,dt \\ &\quad + \gamma_{dD}(t,T)L_d^D(t,T)\,dW_{dD}(t) \\ dL_f^D(t,T) &= \gamma_{fD}(t,T) \\ &\quad \cdot \left(\sigma_{fD}(t,T+\tau) - \rho_{xx}^{fD}(t)\sigma_x(t)\right)L_f^D(t,T)\,dt \\ &\quad + \gamma_{fD}(t,T)L_f^D(t,T)\,dW_{fD}(t) \\ dL_d(t,T) &= \rho_{dD}^{dL}(t)\sigma_{dD}(t,T+\tau)\gamma_{dL}(t,T)L_d(t,T)\,dt \\ &\quad + \gamma_{dL}(t,T)L_d(t,T)\,dW_{dL}(t) \\ dL_f(t,T) &= \gamma_{fL}(t,T) \\ &\quad \cdot \left(\rho_{fD}^{fL}(t)\sigma_{fD}(t,T+\tau) - \rho_{xx}^{fL}(t)\sigma_x(t)\right) \\ &\quad \cdot L_f(t,T)\,dt + \gamma_{fL}(t,T)L_f(t,T)\,dW_{fL}(t) \end{aligned} \tag{25}$$

Remark 4. All the domestically traded assets were checked and it was confirmed that they all satisfy the no arbitrage condition defined in Section 2.3 (iii). This means that all the normalized domestically traded assets discounted using the domestic risk-free money market account as the numeraire were all found to be $\mathbb{Q}_d^D$ martingales.

Proof. See Proof in Appendix A. □

Theorem 5 (the multicurve cross-currency LIBOR market model). *Given the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{Q}_d^D)$ and assuming that the 5-dimensional Wiener process, $W^*(t) = (W_{dD}(t), W_{dL}(t), W_{fD}(t), W_{fL}(t), W_{xx}(t))$, is a vector of correlated Wiener processes such that*

$$d\left[W_i, W_j\right]_t = \begin{cases} \rho_i^j(t)\,dt = \rho_j^i(t)\,dt; & i \neq j \quad i,j \in \{dD, fD, dL, fL, xx\} \\ dt; & i = j \end{cases} \tag{26}$$

then the dynamics of the multicurve cross-currency LIBOR market model under the spot domestic risk neutral martingale measure $\mathbb{Q}_d^D$ is given by

$$\begin{aligned} dB_d^D(t) &= r_d^D(t)B_d^D(t)\,dt \\ dP_d^D(t,T) &= r_d^D(t)P_d^D(t,T)\,dt - \sigma_{dD}(t,T) \\ &\quad \cdot P_d^D(t,T)\,dW_{dD}(t) \\ dL_d^D(t,T) &= \sigma_{dD}(t,T+\tau)\gamma_{dD}(t,T)L_d^D(t,T)\,dt \\ &\quad + \gamma_{dD}(t,T)L_d^D(t,T)\,dW_{dD}(t) \\ dL_f^D(t,T) &= \left(\sigma_{fD}(t,T+\tau) - \rho_{xx}^{fD}(t)\sigma_x(t)\right) \\ &\quad \cdot \gamma_{fD}(t,T)L_f^D(t,T)\,dt + \gamma_{fD}(t,T) \\ &\quad \cdot L_f^D(t,T)\,dW_{fD}(t) \\ dL_d(t,T) &= \rho_{dD}^{dL}(t)\sigma_{dD}(t,T+\tau)\gamma_{dL}(t,T)L_d(t,T)\,dt \\ &\quad + \gamma_{dL}(t,T)L_d(t,T)\,dW_{dL}(t) \\ dL_f(t,T) &= \left(\rho_{fD}^{fL}(t)\sigma_{fD}(t,T+\tau) - \rho_{xx}^{fL}(t)\sigma_x(t)\right) \\ &\quad \cdot \gamma_{fL}(t,T)L_f(t,T)\,dt + \gamma_{fL}(t,T) \\ &\quad \cdot L_f(t,T)\,dW_{fL}(t) \\ dX(t) &= \left(r_d^D(t) - r_f^D(t)\right)X(t)\,dt + \sigma_x(t)\,dW_{xx}(t) \end{aligned} \tag{27}$$

where

$$\sigma_{kD}(t,T) = \sum_{j=1}^{\lfloor \tau^{-1}(T-t)\rfloor} \frac{\tau L_k^D(t,T-j\tau)}{1+\tau L_k^D(t,T-j\tau)}\gamma_{kD}(t,T-j\tau) \tag{28}$$

$$\sigma_{kL}(t,T) = \sum_{j=1}^{\lfloor \tau^{-1}(T-t)\rfloor} \frac{\tau L_k(t,T-j\tau)}{1+\tau L_k(t,T-j\tau)}\gamma_{kL}(t,T-j\tau) \tag{29}$$

where $\lfloor \tau^{-1}(T-t)\rfloor$ denotes the greatest integer that is less than $\tau^{-1}(T-t)$.

Remark 6. Assuming there exists an equivalent risk neutral measure $\mathbb{Q}_d^{D^{T+\tau}} \sim \mathbb{Q}_d^D$ known as the $T+\tau$-forward measure, then, for there to be no arbitrage (as explained in Section 2.3 (v)), all the normalized domestically traded assets discounted using the domestic risk-free zero-coupon bond, $P_d^D(t,T+\tau)$ as the numeraire, must be $\mathbb{Q}_d^{D^{T+\tau}}$ martingales. Hence, as consequence,

$$\begin{aligned} dW_{dD}^{T+\tau}(t) &= dW_{dD}(t) + \sigma_{dD}(t,T+\tau)\,dt \\ dW_{fD}^{T+\tau}(t) &= dW_{fD}(t) + \rho_{dD}^{fD}(t)\sigma_{dD}(t,T+\tau)\,dt \\ dW_{xx}^{T+\tau}(t) &= dW_{xx}(t) + \rho_{xx}^{dD}(t)\sigma_{dD}(t,T+\tau)\,dt \\ dW_{dL}^{T+\tau}(t) &= dW_{dL}(t) + \rho_{dD}^{dL}(t)\sigma_{dD}(t,T+\tau)\,dt \\ dW_{fL}^{T+\tau}(t) &= dW_{fL}(t) + \rho_{dD}^{fL}(t)\sigma_{dD}(t,T+\tau)\,dt \end{aligned} \tag{30}$$

Proof. See proof in Appendix B. □

Remark 7. According to [6], the normalized asset process

$$Z_\Pi(T) = \frac{P_d^L(t,T)}{P_d^L(t,T+\tau)} \tag{31}$$

is a martingale under the $\mathbb{Q}_d^{D^{T+\tau}}$ measure if and only if the risky domestic simply compounded LIBOR satisfies the relationship

$$L_d(t,T) = \frac{1}{\tau}\left(= \frac{P_d^L(t,T)}{P_d^L(t,T+\tau)} - 1\right) \quad (32)$$

However, for $Z_\Pi(T)$ in (26) to be a martingale, then

$$dW_{dL}^{T+\tau}(t) = dW_{dL}^{T}(t) + \sigma_{dL}(t,T+\tau)\,dt \quad (33)$$

and hence to ensure that there are no arbitrage opportunities, this result must be equivalent to the one in (25). This means that

$$\sigma_{dL}(t,T+\tau) = \rho_{dD}^{dL}(t)\,\sigma_{dD}(t,T+\tau) \quad (34)$$

and it is logical to assume that there will always be a positive relationship between the risk-free domestic (or foreign) market and risky domestic (or foreign) market, respectively. Hence

$$\rho_{kD}^{kL} \in [0,1]; \quad k \in \{d,f\} \quad (35)$$

Implying that, for the condition under (29) to hold, then,

$$\sigma_{dL}(t,T+\tau) \le \sigma_{dD}(t,T+\tau) \quad (36)$$

which is a very strict condition. That is why, in this paper, we assume that the equality in (27) is not necessarily true. Hence $Z_\Pi(T)$ on (26) need not be a $\mathbb{Q}_d^{D^{T+\tau}}$ martingale.

Lemma 8. *Let $W^*(t) = (W_{dD}(t), W_{dL}(t), W_{fD}(t), W_{fL}(t), W_{xx}(t))$ be a 5-dimensional $\mathbb{Q}_d^D$ standard Wiener process defined on $(\Omega,\mathcal{F},(\mathcal{F})_{t\ge0},\mathbb{Q}_d^D)$ and Girsanov's kernel, $\phi(t,T+\tau)$, be a 5-dimensional adapted column vector process given by*

$$\phi(t,T+\tau) = -\lambda(t,T+\tau) \quad (37)$$

such that the domestic market price of risk is given by

$$\lambda(t,T+\tau) = \sigma_{dD}(t,T+\tau)\begin{pmatrix} 1 \\ \rho_{dD}^{fD}(t) \\ \rho_{dD}^{dL}(t) \\ \rho_{dD}^{fL}(t) \\ \rho_{xx}^{dD}(t) \end{pmatrix} \quad (38)$$

where

$$\rho_i^j(t) = \rho_j^i(t); \quad i,j \in \{dD, fD, dL, fL, xx\} \qquad 1; \quad i=j \quad (39)$$

and then for a fixed time T, we shall define a process L on $[0,T]$ as the process given by

$$dL(t) = \phi^*(t,T+\tau)\,L(t)\,dW(t) \qquad L(0) = 1 \quad (40)$$

such that $\mathbb{E}^{\mathbb{Q}_d^D}[L_T] = 1$. Therefore, a new probability measure $\mathbb{Q}_d^{D^{T+\tau}}$ on $\mathcal{F}_T$ is now defined by the process

$$L_T = \left.\frac{d\mathbb{Q}_d^{D^{T+\tau}}}{d\mathbb{Q}_d^D}\right|_{\mathcal{F}_T} \quad (41)$$

such that

$$dW(t) = \phi^*(t,T+\tau)\,dt + dW^{T+\tau}(t) \quad (42)$$

where * denotes the transpose and $W^{T+\tau}(t)$ is the 5-dimensional $\mathbb{Q}_d^{T+\tau}$ standard Wiener process defined on $(\Omega,\mathcal{F},(\mathcal{F})_{t\ge0},\mathbb{Q}_d^{D^{T+\tau}})$.

Theorem 9 (the multicurve cross-currency LIBOR market model). *Given the filtered probability space $(\Omega,\mathcal{F},(\mathcal{F})_{t\ge0},\mathbb{Q}_d^{D^{T+\tau}})$ and assuming that the 5-dimensional Wiener process, $W^{T+\tau}(t)$, is a vector of correlated Wiener processes such that*

$$d\left[W_i^{T+\tau}, W_j^{T+\tau}\right]_t = \begin{cases} \rho_i^j(t)\,dt = \rho_j^i(t)\,dt; & i\ne j \quad i,j \in \{dD, fD, dL, fL, xx\} \\ dt; & i=j \end{cases} \quad (43)$$

then the dynamics of the multicurve cross-currency LIBOR market model under the $(T+\tau)$ forward domestic risk neutral martingale measure $\mathbb{Q}_d^{D^{T+\tau}}$ is given by

$$\begin{aligned}
dB_d^D(t) &= r_d^D(t)\,B_d^D(t)\,dt \\
dP_d^D(t,T) &= \left(r_d^D(t) + \sigma_{dD}^2(t,T)\right)P_d^D(t,T)\,dt \\
&\quad - \sigma_{dD}(t,T)\,P_d^D(t,T)\,dW_{dD}^{T+\tau}(t) \\
dL_d^D(t,T) &= \gamma_{dD}(t,T)\,L_d^D(t,T)\,dW_{dD}^{T+\tau}(t) \\
dL_f^D(t,T) &= \Big(\sigma_{fD}(t,T+\tau) - \rho_{xx}^{fD}(t)\,\sigma_x(t) \\
&\quad - \rho_{dD}^{fD}(t)\,\sigma_{dD}(t,T+\tau)\Big)\,\gamma_{fD}(t,T) * L_f^D(t,T)\,dt \\
&\quad + \gamma_{fD}(t,T)\,L_f^D(t,T)\,dW_{fD}^{T+\tau}(t) \\
dL_d(t,T) &= \gamma_{dL}(t,T)\,L_d(t,T)\,dW_{dL}^{T+\tau}(t) \\
dL_f(t,T) &= \Big(\rho_{fD}^{fL}(t)\,\sigma_{fD}(t,T+\tau) - \rho_{xx}^{fL}(t)\,\sigma_x(t) \\
&\quad - \rho_{dD}^{fL}(t)\,\sigma_{dD}(t,T+\tau)\Big) * \gamma_{fL}(t,T)\,L_f(t,T)\,dt \\
&\quad + \gamma_{fL}(t,T)\,L_f(t,T)\,dW_{fL}^{T+\tau}(t) \\
dX(t) &= \left(r_d^D(t) - r_f^D(t) - \rho_{xx}^{dD}(t)\,\sigma_{dD}(t,T+\tau)\right) \\
&\quad \cdot X(t)\,dt + \sigma_x(t)\,dW_{xx}^{T+\tau}(t)
\end{aligned} \quad (44)$$

3. Methodology

In this section, the relevant tools, models, methods, and tests used to achieve the objectives of this study are presented.

3.1. Data. Five datasets were considered in this study. The datasets consisted of the British pound (GBP) overnight and 3-month LIBOR, the overnight and 3-month United States dollar (USD) LIBOR, and the GBP/USD foreign exchange rate. The test datasets consisted of 421 daily trading days' historical data collected from [24–26] for the period between 3/1/2017 and 3/9/2018. The train dataset consisted of 23 daily trading days for the period between 4/9/2018 and 4/10/2018. These datasets only consist of rates recorded on working days in both the United States and the British economy. Hence, it excludes weekends and any public holiday in either economy. We assumed that, in a year, there was an average of 245 working days in both economies.

3.2. Data Analysis Tool. R open source software version 3.1.2 was used in analyzing all the data in this study. Useful packages used were "mass", "stats4", "xts", and "lmtest".

3.3. Parameter Estimation. It was noted that the dynamics of the MCCCLMM were actually forms of the Geometric Brownian motion; hence a brief description of how the model looks like and how the model parameters were estimated is given in this section.

3.3.1. The Geometric Brownian Motion. The Geometric Brownian motion (GBM) solves the stochastic differential equation given by

$$\begin{aligned} dX(t) &= \mu X(t)\,dt + \sigma X(t)\,dW(t) \\ X(0) &= x \end{aligned} \tag{45}$$

where $\mu \in (-\infty, \infty)$ and $\sigma \in (0, \infty]$. The solution of a GBM is given by

$$X(t) = xe^{(\mu-(1/2)\sigma^2)t+\sigma dW(t)} \tag{46}$$

The conditional density function $f(t, y \mid x)$ of a GBM model is log normal with a mean of

$$\mathbb{E}[X(t) \mid X(0) = x] = xe^{\mu t} \tag{47}$$

and a variance of

$$\mathbb{V}[X(t) \mid X(0) = x] = x^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right) \tag{48}$$

Hence

$$\begin{aligned} f(t, y \mid x) &= \frac{1}{y\sqrt{2\pi\sigma^2 t}} \\ &\cdot \exp\left\{-\frac{1}{2\sigma^2 t}\left(\log y - \left(\log x + \left(\mu - \frac{1}{2}\sigma^2 t\right)\right)\right)^2\right\} \end{aligned} \tag{49}$$

3.3.2. Maximum Likelihood Estimation Method. The parameters of the MCCCLMM were estimated using the maximum likelihood estimation method. The method tends to maximize the likelihood function. The maximum likelihood estimate is given by

$$\widehat{\theta} = \max\{\mathscr{L}(\theta; X)\} \tag{50}$$

where $\mathscr{L}(\theta; X)$ is the likelihood function. In R, this is achieved using package "stats4".

3.4. Test of Significance. Individual significance tests performed on the estimated parameters ensure that the fitted parameters are significant. The test hypotheses are

$$\begin{aligned} H_0 &: \theta = 0 \\ H_1 &: \theta \neq 0 \end{aligned} \tag{51}$$

where θ is the parameter estimate under consideration. The null hypothesis is rejected at α level of significance when the p-value is less than α.

3.5. Model Simulation. According to [20], the correlated Wiener process, W, can be simulated by applying Cholesky decomposition as follows:

$$\begin{pmatrix} dW_{dD} \\ dW_{dL} \\ dW_{fD} \\ dW_{fL} \\ dW_{xx} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a & b & 0 & 0 & 0 \\ c & d & e & 0 & 0 \\ f & g & h & i & 0 \\ j & k & l & m & n \end{pmatrix} \begin{pmatrix} dZ_1 \\ dZ_2 \\ dZ_3 \\ dZ_4 \\ dZ_5 \end{pmatrix} \tag{52}$$

where $Z_i;\ i \in \{1, 2, 3, 4, 5\}$ are independent standard normal variables.

$$\begin{aligned} a &= \rho_{dD}^{dL}; \\ b &= \sqrt{1 - a^2}; \\ c &= \rho_{dD}^{fD} \\ d &= \frac{\rho_{dD}^{fD} - ac}{b}; \\ e &= sqrt1 - c^2 - d^2; \\ f &= \frac{\rho_{dD}^{fL} - ac}{b} \\ g &= \frac{\rho_{dL}^{fL} - af}{b}; \\ h &= \frac{\rho_{fD}^{fL} - cf - dg}{e}; \\ i &= \sqrt{1 - f^2 - g^2 - h^2} \\ j &= \rho_{xx}^{dD}; \\ k &= \frac{\rho_{xx}^{dL} - aj}{b}; \\ l &= \frac{\rho_{xx}^{fD} - cj - dk}{e} \\ m &= \frac{\rho_{xx}^{fL} - fj - gk - hl}{i}; \\ n &= \sqrt{1 - j^2 - k^2 - l^2 - m^2} \end{aligned} \tag{53}$$

The Euler Maruyama discretization scheme [27] was then used to simulate the fitted MCCCLMM model.

4. Numerical Results

The MCCCLMM model was fitted to real world data and the results were as recorded in this section.

4.1. Data Description. The data used in this study consisted of 421 trading days' data from 3rd January 2017 to 3rd September 2018. The data consisted of the overnight and 3-month GBP and USD LIBOR term structures obtained from [24, 25] and the GBP/USD foreign exchange rate obtained from [26]. The descriptive statistics of the various sets of the data were recorded in Table 1.

4.2. Parameter Estimation. The parameter estimates of the various dynamics of the multicurve cross-currency LIBOR market model were estimated from the 421 points of the test data via the maximum likelihood estimation (MLE) method using "stats4" package in R. The parameter estimates, their standard errors, and p values were rounded off to the nearest 5 decimal places. The results were as recorded in Table 2.

From Table 2, it can be seen from the respective p-values that all the parameter estimates are significant at 5% level of significance. Also, according to [28], if the standard errors of our estimates are less than 5%, then it is said that the calibration procedure was successful.

4.3. The Simulation Process of the MCCCLMM. The correlation matrix of our observed data was found to be

$$\Sigma = \begin{pmatrix} 1.0000 & 0.9159 & 0.8178 & 0.8495 & -0.5464 \\ 0.9159 & 1.0000 & 0.8625 & 0.9296 & -0.5180 \\ 0.8178 & 0.8625 & 1.0000 & 0.9545 & -0.6154 \\ 0.8495 & 0.9296 & 0.9545 & 1.0000 & -0.6331 \\ -0.5464 & -0.5180 & -0.6154 & -0.6332 & 1.0000 \end{pmatrix} \quad (54)$$

The correlated Wiener process was then estimated as

$$\begin{pmatrix} dW_{dD}(t) \\ dW_{dL}(t) \\ dW_{fD}(t) \\ dW_{fL}(t) \\ dW_{xx}(t) \end{pmatrix} = \sqrt{\Delta t} \begin{pmatrix} 1.0000 & 0.0000 & 0.00000 & 0.0000 & 0.0000 \\ 0.9157 & 0.4015 & 0.0000 & 0.0000 & 0.0000 \\ 0.8178 & 0.2828 & 0.5013 & 0.0000 & 0.0000 \\ 0.8495 & 0.3775 & 0.3055 & 0.2064 & 0.0000 \\ -0.5464 & -0.0439 & -0.3116 & -0.2777 & 0.7248 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \end{pmatrix} \quad (55)$$

where $Z_i;\ i \in \{1, 2, 3, 4, 5\}$ are independent standard normal random variables and $\Delta t = 1/245$. The overnight and the 3M USD LIBOR path, GBP LIBOR path, and the GBP/USD foreign exchange rate path were simulated and their results were as recorded in Sections 4.3.1–4.3.4.

4.3.1. Simulation of the Overnight USD LIBOR. 1000 simulations of the overnight USD LIBOR term structure were done on the train dataset and the descriptive statistics of the simulated path was summarized in Table 3.

The visual plot of the actual path, the 1000 simulated paths, the mean simulated path, and the 95% confidence intervals of the simulated paths were as seen in Figure 3.

It is observed that all the data points lie within the 95% confidence interval of the simulated paths.

4.3.2. Simulation of the 3-Month USD LIBOR. 1000 simulations of the 3-month USD LIBOR term structure were done on the train dataset and the descriptive statistics of the simulated path was summarized in Table 4.

The visual plot of the actual path, the 1000 simulated paths, the mean simulated path, and the 95% confidence intervals of the simulated paths were as seen in Figure 4.

It is observed that all the data points lie within the 95% confidence interval of the simulated paths.

4.3.3. Simulation of the Overnight GBP LIBOR. 1000 simulations of the GBP overnight LIBOR term structure were done on the train dataset and the descriptive statistics of the simulated path was summarized in Table 5.

The visual plot of the actual path, the 1000 simulated paths, the mean simulated path, and the 95% confidence intervals of the simulated paths were as seen in Figure 6.

It is observed that all the data points lie within the 95% confidence interval of the simulated paths.

4.3.4. Simulation of the GBP/USD Foreign Exchange Rate. 1000 simulations of the GBP/USD Foreign exchange rate term structure were done on the train dataset and the descriptive statistics of the simulated path was summarized in Table 7.

The visual plot of the actual path, the 1000 simulated paths, the mean simulated path, and the 95% confidence intervals of the simulated paths were as seen in Figure 5.

It is observed that all the data points lie within the 95% confidence interval of the simulated paths.

4.3.5. Simulation of the 3 Month GBP LIBOR. 1000 simulations of the GBP 3-month LIBOR term structure were done on the train dataset and the descriptive statistics of the simulated path was summarized in Table 6.

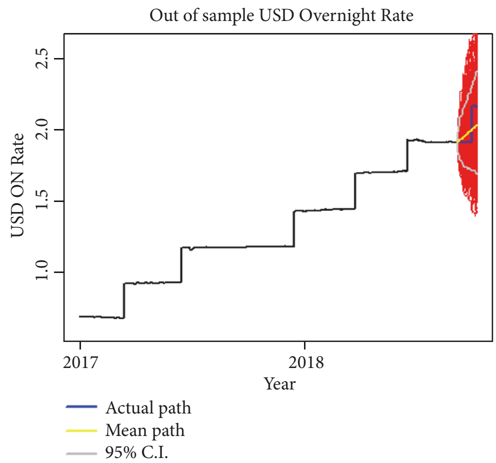

Figure 3: Visual Plot of the Simulated USD ON LIBOR.

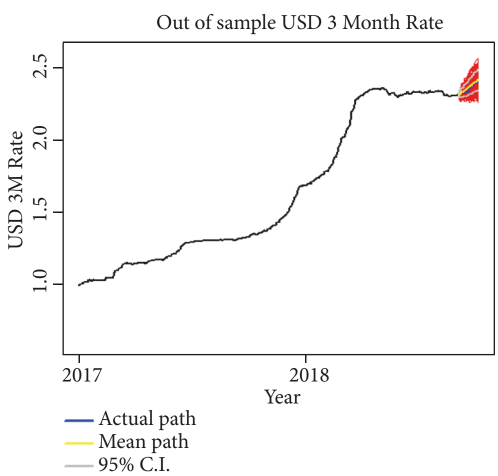

Figure 4: Visual Plot of the Simulated USD 3M LIBOR.

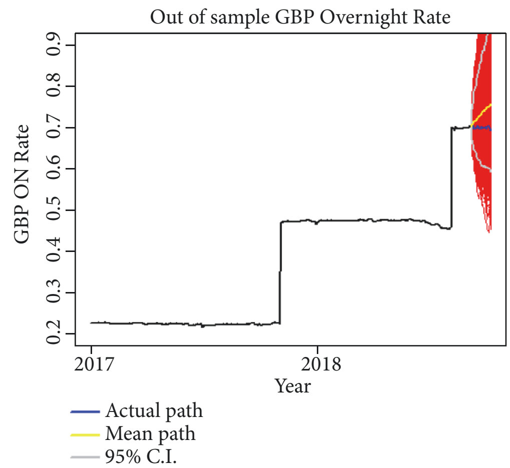

Figure 5: Visual Plot of the Simulated GBP ON LIBOR.

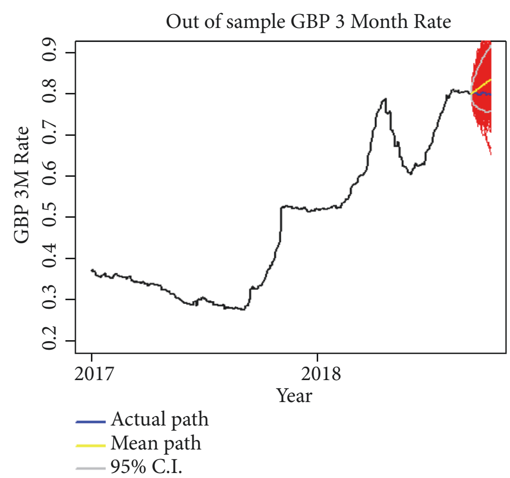

Figure 6: Visual Plot of the Simulated GBP 3M LIBOR.

Table 1: Descriptive Statistics of the Data.

Data	Mean	Std. Dev
ON GBP LIBOR	0.69903	0.00296
3M GBP LIBOR	0.80070	0.00201
ON USD LIBOR	1.98410	0.11364
3M USD LIBOR	2.36071	0.03181
GBP/USD FX Rate	0.76606	0.00596

Table 2: MCCLMM Parameter Estimates.

Parameter	Estimate	Std. Error	P Value
σ_x	0.08355	0.00288	0.00000
σ_{dD}	1.64282	0.76483	0.03172
σ_{fD}	1.83581	0.76547	0.01647
γ_{dD}	0.47450	0.01636	0.00000
γ_{dL}	0.19539	0.00674	0.00000
γ_{fD}	0.35744	0.01232	0.00000
γ_{fL}	0.06692	0.00229	0.00000

Table 3: Descriptive Statistics of the Simulated Overnight USD LIBOR.

Data	Mean	Std. Dev
Simulated ON USD LIBOR	1.9365	0.0688

Table 4: Descriptive Statistics of the Simulated 3 Month USD LIBOR.

Data	Mean	Std. Dev
Simulated 3M USD LIBOR	2.3389	0.0201

Table 5: Descriptive Statistics of the Simulated Overnight GBP LIBOR.

Data	Mean	Std. Dev
Simulated ON GBP LIBOR	0.70610	0.02101

The visual plot of the actual path, the 1000 simulated paths, the mean simulated path, and the 95% confidence intervals of the simulated paths were as seen in Figure 7.

Table 6: Descriptive Statistics of the Simulated 3 Month GBP LIBOR.

Data	Mean	Std. Dev
Simulated 3M GBP LIBOR	0.82660	0.01059

Table 7: Descriptive Statistics of the Simulated GBP/USD FX Rate.

Data	Mean	Std. Dev
Simulated G/USD FX Rate	0.76631	0.00482

Table 8: Mean Error Test Results of the Simulated Models.

Simulated Dataset	MAPE (%)
ON USD LIBOR	7.7663
3M USD LIBOR	3.1415
ON GBP LIBOR	6.8195
3M GBP LIBOR	1.9917
GBP/USD FX RATE	1.7604

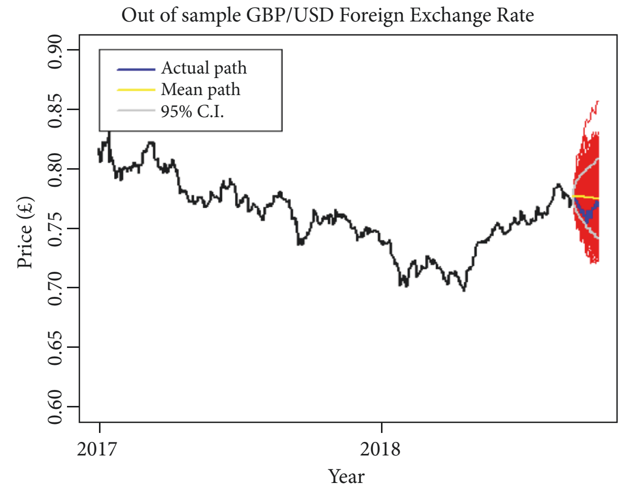

Figure 7: Visual Plot of the Simulated GBP/USD FX Rate.

It is observed that all the data points lie within the 95% confidence interval of the simulated paths.

4.4. Testing the Fitted MCCCLMM Dynamics. The mean absolute percentage error (MAPE) was performed on the difference between the actual and the 1000 simulated paths of the overnight and 3-month USD LIBOR, GBP LIBOR, and the GBP/USD foreign exchange rate and the results were as recorded in Table 8.

According to [29], if $MAPE < 10\%$, then the forecasts are highly accurate. If $10\% \leq MAPE < 20\%$, then the forecasts are good. If $20\% \leq MAPE \leq 50\%$, then the forecasts are reasonable. However, if $MAPE > 50\%$, then the forecasts are inaccurate. Hence, we can conclude that the MCCCLMM produces highly accurate forecasts for the 23 working days' period.

5. Application to Pricing Quanto Caplets and Floorlets

In this section, we illustrate briefly how the dynamics derived under the multicurve cross-currency LIBOR market model can be used to price quanto interest rate derivatives such as quanto caplets and quanto floorlets. A caplet is a call optional type of interest rate derivative where the investor or hedger receives payments if the interest rate exceeds the agreed strike price at maturity. In the same way, a floorlet is a put optional type of interest rate derivative where the hedger receives payment if the interest rate falls below the agreed strike price at maturity. In addition to this, a quanto is a type of derivative whereby the underlying instrument is denominated in one currency but settled in another currency.

We start by assuming that there exists a domestic investor who is interested in hedging against a foreign interest rate risk. We also assume that the investor is more comfortable in using his/her domestic currency in trading and hence all the quanto caplets and floorlet described hereby are priced in terms of the domestic currency. We shall consider a scenario whereby a foreign caplet or floorlet is struck in foreign currency but has to be converted into domestic currency using either a fixed exchange rate or a floating exchange rate.

5.1. Fixed Exchange Rate. Consider a domestic investor who wishes to buy a caplet or floorlet contract struck in foreign currency at a foreign strike price of K_f where the underlying is the risky foreign LIBOR forward rate, $L_f(t,T)$. If the seller of the option fixes the exchange rate at an agreed value say $\widehat{X}$ at the inception of the contract, then the payoff of this contract at time T expressed in domestic currency will be given by

$$\Phi_d(T) = \widehat{X} N_f \left[\omega * \left(L_f(T,T) - K_f\right)\right]^+ \tag{56}$$

where N_f is the pricipal of the option expressed in units of foreign currency, $\widehat{X}$ is the fixed exchange rate agreed upon at the inception of the contract expressed as the ratio of domestic currency to one unit of foreign currency, and ω is a binary operator such that

$$\omega = \begin{cases} +1 & \text{if it is a caplet contract} \\ -1 & \text{if it is a floorlet contract} \end{cases} \tag{57}$$

Let the value of the foreign caplet or floorlet at time t expressed in domestic currency be denoted by $V_d(\omega, t, K_f, L_f)$. Therefore, since a fixed exchange rate is considered then it should be noted that the dynamics of the foreign risky forward LIBOR, $L_f(t,T)$, in this case will be valued in the domestic economy. Hence under the domestic risk neutral $T+\tau$ forward measure $\mathbb{Q}_d^{D^{T+\tau}}$,

$$\begin{aligned} \frac{dL_f(t,T)}{L_f(t,T)} &= \left(\rho_{fD}^{fL}(t)\,\sigma_{fD}(t,T+\tau) - \rho_{xx}^{fL}(t)\,\sigma_x(t) \right. \\ &\quad \left. - \rho_{dD}^{fL}(t)\,\sigma_{dD}(t,T)\right)\gamma_{fL}(t,T)\,dt \\ &\quad + \gamma_{fL}(t,T)\,dW_{fL}^{T+\tau}(t) \end{aligned} \tag{58}$$

Hence the pricing boundary value problem on $[0,T]\mathrm{x}\mathbb{R}$ used to price such a contract will be given by

$$\frac{\partial V_d(t,L_f)}{\partial t}+\left(\rho_{fD}^{fL}(t)\sigma_{fD}(t,T+\tau)-\rho_{xx}^{fL}(t)\sigma_x(t)-\rho_{dD}^{fL}(t)\sigma_{dD}(t,T+\tau)\right)*\gamma_{fL}(t,T)*L_f(t,T)\frac{\partial V_d(t,L_f)}{\partial L_f(t,T)}+\frac{1}{2}\gamma_{fL}^2(t,T)\cdot L_f^2(t,T)\frac{\partial^2 V_d(t,L_f)}{\partial L_f^2(t,T)}=r_d^D(t)$$
$$V_d(t,L_f)V_d(T,L_f)=\Phi_d(T) \tag{59}$$

The value at time $t\le T$ of the foreign caplet or floorlet struck in foreign currency but expressed in terms of domestic currency to one unit of foreign currency given that a fixed exchange rate is considered will therefore be given by

$$\begin{aligned}V_d(t,L_f)&=e^{-\int_t^T r_d^D(s)ds}\mathbb{E}^{\mathbb{Q}_d^{D^{T+\tau}}}\left[\Phi_d(T)\mid\mathcal{F}_t\right]\\&=P_d^D(t,T)\mathbb{E}^{\mathbb{Q}_d^{D^{T+\tau}}}\left[\Phi_d(T)\mid\mathcal{F}_t\right]\\&=\omega\widehat{X}N_fP_d^D(t,T)\\&\cdot\left(L_f(t,T)e^{\int_t^T\mu_{fL}^d(u,T,T+\tau)du}N(\omega*d_{1f})\right.\\&\left.-K_fN(\omega*d_{2f})\right)\end{aligned} \tag{60}$$

where

$$\begin{aligned}\mu_{fL}^d(t,T,T+\tau)&=\left(\rho_{fD}^{fL}(t)\sigma_{fD}(t,T+\tau)-\rho_{xx}^{fL}(t)\sigma_x(t)-\rho_{dD}^{fL}(t)\sigma_{dD}(t,T+\tau)\right)\gamma_{fL}(t,T)\\d_{1f}&=\frac{1}{\nu(t,T)}\left[\ln\left(\frac{L_f(t,T)}{K_f}\right)+\frac{1}{2}\nu^2(t,T)+\int_t^T\mu_{fL}(u,T,T+\tau)du\right]\\d_{2f}&=d_{1f}-\nu(t,T)\\\nu^2(t,T)&=\int_t^T\gamma_{fL}^2(u,T)du\end{aligned} \tag{61}$$

and $N(\cdot)$ is the cumulative standard normal distribution.

The advantage of such a contract is that it will shield the domestic investor from the risk of exposure to exchange rate risk.

5.2. Floating Exchange Rate. Consider a domestic investor who wishes to buy a caplet or floorlet contract struck in foreign currency at a foreign strike price of K_f where the underlying is the risky foreign LIBOR forward rate, $L_f(t,T)$. If the seller of the option assumes that the exchange rate that will be considered will be the spot exchange rate at maturity, X(T), then the payoff of this contract at time T expressed in domestic currency will be given by

$$\Phi_f(T)=X(T)N_f\left[\omega*\left(L_f(T,T)-K_f\right)\right]^+ \tag{62}$$

where N_f is the pricipal of the option expressed in units of foreign currency, $X(T)$ is the spot foreign exchange rate at maturity of the contract expressed as the ratio of domestic currency to one unit of foreign currency, and ω is a binary operator such that

$$\omega=\begin{cases}+1 & \text{if it is a caplet contract}\\-1 & \text{if it is a floorlet contract}\end{cases} \tag{63}$$

Let the value of the foreign caplet or floorlet at time t expressed in domestic currency be denoted by $V_d(\omega,t,K_f,L_f)$. Therefore, since a floating exchange rate is considered, then it should be noted that the position is unhedged meaning that the contract is exposed to foreign exchange rate risk. This means that the dynamics of the foreign risky forward LIBOR, $L_f(t,T)$, in this case will rely directly on the foreign economy. Hence under the foreign risk neutral $T+\tau$ forward measure $\mathbb{Q}_f^{D^{T+\tau}}$,

$$\frac{dL_f(t,T)}{L_f(t,T)}=\gamma_{fL}(t,T)\,dW_{fL}^{T+\tau}(t) \tag{64}$$

Hence the pricing boundary value problem on $[0,T]\mathrm{x}\mathbb{R}$ used to price such a contract will be given by

$$\begin{aligned}&\frac{\partial V_d(t,L_f)}{\partial t}+\frac{1}{2}\gamma_{fL}^2(t,T)L_f^2(t,T)\frac{\partial^2 V_d(t,L_f)}{\partial L_f^2(t,T)}-r_f^D(t)V_d(t,L_f)=0\\&V_d(T,L_f)=\Phi_f(T)\end{aligned} \tag{65}$$

The value at time $t\le T$ of the foreign caplet or floorlet struck in foreign currency but expressed in terms of domestic currency to one unit of foreign currency given that a fixed exchange rate is considered will therefore be given by

$$\begin{aligned}V_d(t,L_f)&=e^{-\int_t^T r_f^D(s)ds}\mathbb{E}^{\mathbb{Q}_f^{D^{T+\tau}}}\left[\Phi_f(T)\mid\mathcal{F}_t\right]\\&=P_f^D(t,T)\mathbb{E}^{\mathbb{Q}_f^{D^{T+\tau}}}\left[\Phi_f(T)\mid\mathcal{F}_t\right]=\omega X(t)\\&\cdot N_fP_f^D(t,T)\\&\cdot\left[L_f(t,T)N(\omega*d_{1g})-K_fN(\omega*d_{2g})\right]\end{aligned} \tag{66}$$

where

$$d_{1g} = \frac{1}{v(t,T)}\left[\ln\left(\frac{L_f(t,T)}{K_f}\right) + \frac{1}{2}v^2(t,T)\right]$$
$$d_{2g} = d_{1f} - v(t,T) \tag{67}$$
$$v^2(t,T) = \int_t^T \gamma_{fL}^2(u,T)\,du$$

and $N(\cdot)$ is the cumulative standard normal distribution. It should be noted that this type of contract is directly affected by exchange rate movements. The advantage of such a contract is that if the exchange rate moves upwards (or downwards for the floorlet option), then the domestic investor is set to make a profit. However, if the exchange rate moves downwards (or upwards for the floorlet option), then the payout from the option is set to reduce.

6. Conclusion

This study extends the concepts by [20, 21] into the multiple curve cross-currency setting. It focused on constructing a model that can be used to model the simply compounded forward rates of both the domestic and foreign markets under the domestic risk neutral probability measure. The study assumed that interest rates are strictly positive, hence the choice of lognormal type of models. However, in recent times, negative IBOR rates have been recorded. Therefore, this study can be extended to include models that incorporate negative interest rates.

Model parameters were estimated using 421 trading days' datasets of the GBP overnight and 3-month LIBOR, USD overnight and 3-month LIBOR, and the GBP/USD foreign exchange rate data for the period between 3rd January, 2017, and 3rd September, 2018. The estimated parameters were then used to simulate 1,000 sample paths of 23 trading days out of sample estimates from 4th September, 2018, to 4th October, 2018. From the mean absolute percentage errors (MAPE) calculated, it was seen that the MCCCLMM produces accurate results for this period.

An illustration of how quanto optional interest rate derivatives such as the quanto caplets and floorlets can be valued under the MCCCLMM was also done. However, this should not limit the research on valuing other types of derivatives as the derived MCCCLMM is a robust model that can be used to price numerous interest rate derivatives.

In addition to this, the derived MCCCLMM model is a relatively new model yet to be tested on various term structures or interest rate derivatives and hence pricing performance tests can still be done on it to ascertain its accuracy when applied to more real world data. This will help illustrate how the model can be effectively used in pricing various interest rate derivatives including quantos.

Appendix

A. Proof That the No Arbitrage Condition in Section 2.3 (iii) Is Satisfied

(1)

$$Z_{dD}(t,T) = \frac{P_d^D(t,T)}{B_d^D(t)} \tag{A.1}$$

is a $\mathbb{Q}^D$ martingale.

Proof.

$$\frac{dZ_{dD}(t,T)}{Z_{dD}(t,T)} = r_d^D(t)\,dt - \sigma_{dD}(t,T)\,dW_{dD}(t) - r_d^D(t)\,dt = -\sigma_{dD}(t,T)\,dW_{dD}(t) \tag{A.2}$$

which is a martingale under the $\mathbb{Q}^D$ measure. □

(2)

$$Z_{fD}(t,T) = \frac{P_f^{*D}(t,T)}{B_d^D(t)} \tag{A.3}$$

is a $\mathbb{Q}^D$ martingale.

Proof.

$$\frac{dZ_{fD}(t,T)}{Z_{fD}(t,T)} = r_d^D(t)\,dt - \sigma_{fD}(t,T)\,dW_{fD}(t) + \sigma_x(t)\,dW_{xx}(t) - r_d^D(t)\,dt = \sigma_x(t)\,dW_{xx}(t) - \sigma_{fD}(t,T)\,dW_{fD}(t) \tag{A.4}$$

which is a martingale under the $\mathbb{Q}^D$ measure. □

(3)

$$Z_{dDL}(t,T) = \frac{P_d^D(t,T+\tau)\,L_d^D(t,T)}{B_d^D(t)} \tag{A.5}$$

is a $\mathbb{Q}^D$ martingale.

Proof.

$$\begin{aligned}\frac{dZ_{dDL}(t,T)}{Z_{dDL}(t,T)} &= r_d^D(t)\,dt - \sigma_{dD}(t,T+\tau)\,dW_{dD}(t) + \sigma_{dD}(t,T+\tau)\,\gamma_{dD}(t,T)\,dt + \gamma_{dD}(t,T)\,dW_{dD}(t) \\ &\quad - \sigma_{dD}(t,T+\tau)\,\gamma_{dD}(t,T)\,dt - r_d^D(t)\,dt \\ &= \left(\gamma_{dD}(t,T) - \sigma_{dD}(t,T+\tau)\right)dW_{dD}(t)\end{aligned} \tag{A.6}$$

which is a martingale under the $\mathbb{Q}^D$ measure. □

(4)

$$Z_{fDL}(t,T) = \frac{P_f^{*D}(t,T+\tau)\, L_f^D(t,T)}{B_d^D(t)} \tag{A.7}$$

is a $\mathbb{Q}^D$ martingale.

Proof.

$$\begin{aligned}\frac{dZ_{fDL}(t,T)}{Z_{fDL}(t,T)} &= r_d^D(t)\,dt - \sigma_{fD}(t,T+\tau)\,dW_{fD}(t) \\ &\quad + \sigma_x(t)\,dW_{xx}(t) + \sigma_{fD}(t,T+\tau)\,\gamma_{fD}(t,T)\,dt \\ &\quad - \rho_{xx}^{fD}(t)\,\sigma_x(t)\,\gamma_{fD}(t,T)\,dt \\ &\quad + \gamma_{fL}(t,T)\,dW_{fD}(t) \\ &\quad - \sigma_{fD}(t,T+\tau)\,\gamma_{fD}(t,T)\,dt \\ &\quad + \rho_{xx}^{fD}(t)\,\sigma_x(t)\,\gamma_{fD}(t,T)\,dt - r_d^D(t)\,dt \\ &= \sigma_x(t)\,dW_{xx}(t) \\ &\quad + \left(\gamma_{fD}(t,T) - \sigma_{fD}(t,T+\tau)\right) dW_{fD}(t)\end{aligned} \tag{A.8}$$

which is a martingale under the $\mathbb{Q}^D$ measure. □

(5)

$$Z_{dLL}(t,T) = \frac{P_d^D(t,T+\tau)\, L_d(t,T)}{B_d^D(t)} \tag{A.9}$$

is a $\mathbb{Q}^D$ martingale.

Proof.

$$\begin{aligned}\frac{dZ_{dLL}(t,T)}{Z_{dLL}(t,T)} &= r_d^D(t)\,dt - \sigma_{dD}(t,T+\tau)\,dW_{dD}(t) \\ &\quad + \rho_{dD}^{dL}(t)\,\sigma_{dD}(t,T+\tau)\,\gamma_{dL}(t,T)\,dt \\ &\quad + \gamma_{dL}(t,T)\,dW_{dL}(t) \\ &\quad - \rho_{dD}^{dL}(t)\,\sigma_{dD}(t,T+\tau)\,\gamma_{dL}(t,T)\,dt \\ &\quad - r_d^D(t)\,dt \\ &= \gamma_{dL}(t,T)\,dW_{dL}(t) \\ &\quad - \sigma_{dD}(t,T+\tau)\,dW_{dD}(t)\end{aligned} \tag{A.10}$$

which is a martingale under the $\mathbb{Q}^D$ measure. □

(6)

$$Z_{fLL}(t,T) = \frac{P_f^{*D}(t,T+\tau)\, L_f^D(t,T)}{B_d^D(t)} \tag{A.11}$$

is a $\mathbb{Q}^D$ martingale.

Proof.

$$\begin{aligned}\frac{dZ_{fLL}(t,T)}{Z_{fLL}(t,T)} &= r_d^D(t)\,dt - \sigma_{fD}(t,T+\tau)\,dW_{fD}(t) \\ &\quad + \sigma_x(t)\,dW_{xx}(t) \\ &\quad + \rho_{fD}^{fL}(t)\,\sigma_{fD}(t,T+\tau)\,\gamma_{fL}(t,T)\,dt \\ &\quad - \rho_{xx}^{fL}(t)\,\sigma_x(t)\,\gamma_{fL}(t,T)\,dt \\ &\quad + \gamma_{fL}(t,T)\,dW_{fL}(t) \\ &\quad - \rho_{fD}^{fL}(t)\,\sigma_{fD}(t,T+\tau)\,\gamma_{fL}(t,T)\,dt \\ &\quad + \rho_{xx}^{fL}(t)\,\sigma_x(t)\,\gamma_{fL}(t,T)\,dt \\ &\quad - r_d^D(t)\,dt \\ &= \sigma_x(t)\,dW_{xx}(t) + \gamma_{fL}(t,T)\,dW_{fL}(t) \\ &\quad - \sigma_{fD}(t,T+\tau)\,dW_{fD}(t)\end{aligned} \tag{A.12}$$

which is a martingale under the $\mathbb{Q}^D$ measure. □

B. Proof That the No Arbitrage Condition in Section 2.3 (v) Is Satisfied

(1)

$$Z_{dD}(t,T+\tau) = \frac{P_d^D(t,T)}{P_d^D(t,T+\tau)} \tag{B.1}$$

is a $\mathbb{Q}^{D^{T+\tau}}$ martingale.

Proof.

$$\begin{aligned}\frac{dZ_{dD}(t,T+\tau)}{Z_{dD}(t,T+\tau)} &= r_d^D(t)\,dt - \sigma_{dD}(t,T)\,dW_{dD}(t) \\ &\quad - r_d^D(t)\,dt + \sigma_{dD}(t,T+\tau)\,dW_{dD}(t) - \sigma_{dD}(t,T) \\ &\quad \cdot \sigma_{dD}(t,T+\tau)\,dt + \left(\sigma_{dD}(t,T+\tau)\right)^2 dt \\ &= \sigma_{dD}(t,T+\tau)\left(\sigma_{dD}(t,T+\tau) - \sigma_{dD}(t,T)\right) dt \\ &\quad + \left(\sigma_{dD}(t,T+\tau) - \sigma_{dD}(t,T)\right) dW_{dD}(t) \\ &= \left(\sigma_{dD}(t,T+\tau) - \sigma_{dD}(t,T)\right) \\ &\quad \cdot \left[dW_{dD}(t) + \sigma_{dD}(t,T+\tau)\,dt\right] \\ &= \left(\sigma_{dD}(t,T+\tau) - \sigma_{dD}(t,T)\right) dW_{dD}^{T+\tau}(t)\end{aligned} \tag{B.2}$$

which is a martingale under the $\mathbb{Q}^{D^{T+\tau}}$ measure, where

$$dW_{dD}^{T+\tau}(t) = dW_{dD}(t) + \sigma_{dD}(t,T+\tau)\,dt \tag{B.3}$$

□

(2)

$$Z_{fD}(t,T+\tau) = \frac{P_f^{*D}(t,T)}{P_d^D(t,T+\tau)} \tag{B.4}$$

is a $\mathbb{Q}^{D^{T+\tau}}$ martingale.

Proof.

$$\begin{aligned}
\frac{dZ_{fD}(t,T+\tau)}{Z_{fD}(t,T+\tau)} &= r_d^D(t)\,dt - \sigma_{fD}(t,T)\,dW_{fD}(t) \\
&\quad + \sigma_x(t)\,dW_{xx}(t) - r_d^D(t)\,dt \\
&\quad + \sigma_{dD}(t,T+\tau)\,dW_{dD}(t) \\
&\quad + \rho_{dD}^{fD}(t)\,\sigma_{fD}(t,T)\,\sigma_{dD}(t,T+\tau)\,dt \\
&\quad - \rho_{xx}^{dD}(t)\,\sigma_x(t)\,\sigma_{dD}(t,T+\tau)\,dt \\
&\quad + (\sigma_{dD}(t,T+\tau))^2\,dt \\
&= \sigma_x(t)\left[dW_{xx}(t) - \rho_{xx}^{dD}(t)\,\sigma_{dD}(t,T+\tau)\,dt\right] \\
&\quad - \sigma_{fD}(t,T)\left[dW_{fD}(t) + \rho_{dD}^{fD}(t)\,\sigma_{dD}(t,T+\tau)\,dt\right] \\
&\quad + \sigma_{dD}(t,T+\tau)\left[dW_{dD}(t) + \sigma_{dD}(t,T+\tau)\,dt\right] \\
&= \sigma_x(t)\,dW_{xx}^{T+\tau}(t) - \sigma_{fD}(t,T)\,dW_{fD}^{T+\tau}(t) \\
&\quad + \sigma_{dD}(t,T+\tau)\,dW_{dD}^{T+\tau}(t)
\end{aligned} \tag{B.5}$$

which is a martingale under the $\mathbb{Q}^{D^{T+\tau}}$ measure, where

$$dW_{fD}^{T+\tau}(t) = dW_{fD}(t) + \rho_{dD}^{fD}(t)\,\sigma_{dD}(t,T+\tau)\,dt \tag{B.6}$$

$$dW_{xx}^{T+\tau}(t) = dW_{xx}(t) + \rho_{xx}^{dD}(t)\,\sigma_{dD}(t,T+\tau)\,dt \tag{B.7}$$

□

(3)

$$Z_{dDL}(t,T+\tau) = \frac{P_d^D(t,T+\tau)\,L_d^D(t,T)}{P_d^D(t,T+\tau)} \tag{B.8}$$

is a $\mathbb{Q}^{D^{t+\tau}}$ martingale.

Proof.

$$\begin{aligned}
&\frac{dZ_{dDL}(t,T+\tau)}{Z_{dDL}(t,T+\tau)} \\
&\quad = r_d^D(t)\,dt \\
&\qquad + (\gamma_{dD}(t,T) - \sigma_{dD}(t,T+\tau))\,dW_{dD}(t) \\
&\qquad - r_d^D(t) + \sigma_{dD}(t,T+\tau)\,dW_{dD}(t) \\
&\qquad + \sigma_{dD}(t,T+\tau)\,(\gamma_{dD}(t,T) - \sigma_{dD}(t,T+\tau))\,dt \\
&\qquad + (\sigma_{dD}(t,T+\tau))^2\,dt \\
&\quad = \gamma_{dD}(t,T)\left[dW_{dD}(t) + \sigma_{dD}(t,T+\tau)\,dt\right] \\
&\quad = \gamma_{dD}(t,T)\,dW_{dD}^{T+\tau}(t)
\end{aligned} \tag{B.9}$$

which is a martingale under the $\mathbb{Q}^{D^{T+\tau}}$ measure. □

(4)

$$Z_{fDL}(t,T+\tau) = \frac{P_f^{*D}(t,T+\tau)\,L_f^D(t,T)}{P_d^D(t,T+\tau)} \tag{B.10}$$

is a $\mathbb{Q}^{D^{T+\tau}}$ martingale.

Proof.

$$\begin{aligned}
\frac{dZ_{fDL}(t,T+\tau)}{Z_{fDL}(t,T+\tau)} &= r_d^D(t) + \sigma_x(t)\,dW_{xx}(t) \\
&\quad + \left(\gamma_{fD}(t,T) - \sigma_{fD}(t,T+\tau)\right)dW_{fD}(t) \\
&\quad - r_d^D(t)\,dt + \sigma_{dD}(t,T+\tau)\,dW_{dD}(t) \\
&\quad + (\sigma_{dD}(t,T+\tau))^2\,dt + \rho_{dD}^{fD}(t)\,\sigma_{dD}(t,T+\tau) \\
&\quad \cdot \left(\gamma_{fD}(t,T) - \sigma_{fD}(t,T+\tau)\right)dt + \rho_{xx}^{dD}(t)\,\sigma_x(t) \\
&\quad \cdot \sigma_{dD}(t,T+\tau)\,dt = \sigma_x(t) \\
&\quad \cdot \left[dW_{xx}(t) + \rho_{xx}^{dD}(t)\,\sigma_{dD}(t,T+\tau)\,dt\right] \\
&\quad + \sigma_{dD}(t,T+\tau)\left[dW_{dD}(t) + \sigma_{dD}(t,T+\tau)\,dt\right] \\
&\quad - \left(\gamma_{fD}(t,T) - \sigma_{fD}(t,T+\tau)\right) \\
&\quad \cdot \left[dW_{fD}(t) + \rho_{dD}^{fD}(t)\,\sigma_{dD}(t,T+\tau)\,dt\right] \\
&\quad = \sigma_x(t)\,dW_{xx}^{T+\tau}(t) \\
&\quad + \left(\gamma_{fD}(t,T) - \sigma_{fD}(t,T+\tau)\right)dW_{fD}^{T+\tau}(t) \\
&\quad + \sigma_{dD}(t,T+\tau)\,dW_{dD}^{T+\tau}(t)
\end{aligned} \tag{B.11}$$

which is a martingale under the $\mathbb{Q}^D$ measure. □

(5)

$$Z_{dLL}(t,T+\tau) = \frac{P_d^D(t,T+\tau)\,L_d(t,T)}{P_d^D(t,T+\tau)} \tag{B.12}$$

is a $\mathbb{Q}^{D^{T+\tau}}$ martingale.

Proof.

$$\begin{aligned}
&\frac{dZ_{dLL}(t,T+\tau)}{Z_{dLL}(t,T+\tau)} \\
&\quad = r_d^D(t)\,dt + \gamma_{dL}(t,T)\,dW_{dL}(t) \\
&\qquad - \sigma_{dD}(t,T+\tau)\,dW_{dD}(t) - r_d^D(t)\,dt \\
&\qquad + \sigma_{dD}(t,T+\tau)\,dW_{dD}(t) \\
&\qquad + \rho_{dD}^{dL}(t)\,\gamma_{dL}(t,T)\,\sigma_{dD}(t,T+\tau)\,dt \\
&\qquad - (\sigma_{dD}(t,T+\tau))^2\,dt + (\sigma_{dD}(t,T+\tau))^2\,dt
\end{aligned}$$

$$= \gamma_{dL}(t,T)\left[dW_{dL}(t) + \rho_{dD}^{dL}(t)\,\sigma_{dD}(t,T+\tau)\,dt\right]$$
$$= \gamma_{dL}(t,T)\,dW_{dL}^{T+\tau}(t) \quad \text{(B.13)}$$

which is a martingale under the $\mathbb{Q}^{D^{T+\tau}}$ measure, where

$$dW_{dL}^{T+\tau}(t) = dW_{dL}(t) + \rho_{dD}^{dL}(t)\,\sigma_{dD}(t,T+\tau)\,dt \quad \text{(B.14)}$$

□

(6)

$$Z_{fLL}(t,T) = \frac{P_f^{*D}(t,T+\tau)\,L_f^D(t,T)}{P_d^D(t,T+\tau)} \quad \text{(B.15)}$$

is a $\mathbb{Q}^{D^{T+\tau}}$ martingale.

Proof.

$$\begin{aligned}
\frac{dZ_{fLL}(t,T+\tau)}{Z_{fLL}(t,T+\tau)} &= r_d^D(t)\,dt + \sigma_x(t)\,dW_{xx}(t) \\
&+ \gamma_{fL}(t,T)\,dW_{fL}(t) - \sigma_{fD}(t,T+\tau)\,dW_{fD}(t) \\
&- r_d^D(t)\,dt + \sigma_{dD}(t,T+\tau)\,dW_{dD}(t) + \rho_{xx}^{dD}(t) \\
&\cdot \sigma_x(t)\,\sigma_{dD}(t,T+\tau)\,dt + \rho_{dD}^{fL}(t)\,\gamma_{fL}(t,T) \\
&\cdot \sigma_{dD}(t,T+\tau)\,dt - \rho_{dD}^{fD}\sigma_{fD}(t,T+\tau) \\
&\cdot \sigma_{dD}(t,T+\tau)\,dt + \left(\sigma_{dD}(t,T+\tau)\right)^2 dt = \sigma_x(t) \\
&\cdot \left[dW_{xx}(t) + \rho_{xx}^{dD}(t)\,\sigma_{dD}(t,T+\tau)\,dt\right] \\
&+ \gamma_{fL}(t,T)\left[dW_{fL}(t) + \rho_{dD}^{fL}(t)\,\sigma_{dD}(t,T+\tau)\,dt\right] \\
&- \sigma_{fD}(t,T+\tau) \\
&\cdot \left[dW_{fD}(t) + \rho_{dD}^{fD}(t)\,\sigma_{dD}(t,T+\tau)\right] \\
&+ \sigma_{dD}(t,T+\tau)\left[dW_{dD}(t) + \sigma_{dD}(t,T+\tau)\,dt\right] \\
&= \sigma_x(t)\,dW_{xx}^{T+\tau}(t) + \gamma_{fL}(t,T)\,dW_{fL}^{T+\tau}(t) \\
&- \sigma_{fD}(t,T+\tau)\,dW_{fD}^{T+\tau}(t) \\
&+ \sigma_{dD}(t,T+\tau)\,dW_{dD}^{T+\tau}(t)
\end{aligned} \quad \text{(B.16)}$$

which is a martingale under the $\mathbb{Q}^{D^{T+\tau}}$ measure, □

where

$$dW_{fL}^{T+\tau}(t) = dW_{fL}(t) + \rho_{dD}^{fL}(t)\,\sigma_{dD}(t,T+\tau)\,dt \quad \text{(B.17)}$$

Data Availability

The data used is freely available on the websites as at the date last accessed. See [24–26].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The first author acknowledges financial support from Pan African University Institute of Basic and Applied Sciences Scholarship.

References

[1] J. Cui, F. In, and E. Maharaj, "What drives the Libor–ois spread? evidence from five major currency libor–ois spreads," *International Review of Economics & Finance*, vol. 45, pp. 358–375, 2016.

[2] F. M. Ametrano and M. Bianchetti, "Everything you always wanted to know about multiple interest rate curve bootstrapping but were afraid to ask," *SSRN Electronic Journal*, 82 pages, 2013.

[3] M. Bianchetti and M. Carlicchi, "Interest rates after the credit crunch: multiple curve vanilla derivatives and sabr," *SSRN Electronic Journal*, 2011.

[4] S. Crépey, Z. Grbac, and H.-N. Nguyen, "A multiple-curve HJM model of interbank risk," *Mathematics and Financial Economics*, vol. 6, no. 3, pp. 155–190, 2012.

[5] C. Cuchiero, C. Fontana, and A. Gnoatto, "A general HJM framework for multiple yield curve modelling," *Finance and Stochastics*, vol. 20, no. 2, pp. 267–320, 2016.

[6] M. Grasselli and G. Miglietta, "A flexible spot multiple-curve model," *Quantitative Finance*, vol. 16, no. 10, pp. 1465–1477, 2016.

[7] Z. Grbac, A. Papapantoleon, J. Schoenmakers, and D. Skovmand, "Affine libor models with multiple curves: theory, examples and calibration," *SIAM Journal on Financial Mathematics*, vol. 6, no. 1, pp. 984–1025, 2015.

[8] F. Mercurio, "Interest rates and the credit crunch: new formulas and market models," *SSRN Electronic Journal*, 2009.

[9] F. Mercurio, "Libor market models with stochastic basis," *SSRN Electronic Journal*, 2010.

[10] M. Hess, "An arithmetic pure-jump multi-curve interest rate model," *SSRN Electronic Journal*, 2016.

[11] F. Mercurio, "The present of futures: valuing eurodollar-futures convexity adjustments in a multi-curve world," *SSRN Electronic Journal*, 2017.

[12] M. Bianchetti, "Two curves, one price: pricing & hedging interest rate derivatives decoupling forwarding and discounting yield curves," https://arxiv.org/abs/0905.2770.

[13] M. Henrard, "The irony in the derivatives discounting," *SSRN Electronic Journal*, 2007.

[14] O. Vasicek, "An equilibrium characterization of the term structure," *Journal of Financial Economics*, vol. 5, no. 2, pp. 177–188, 1977.

[15] J. C. Cox, J. Ingersoll, and S. A. Ross, "A theory of the term structure of interest rates," *Econometrica*, vol. 53, no. 2, pp. 321–346, 1985.

[16] J. Hull and A. White, "Pricing interest-rate derivative securities," *Review of Financial Studies* , vol. 3, no. 4, pp. 573–592, 1990.

[17] D. Heath, R. Jarrow, and A. Morton, "Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation," *Econometrica: Journal of the Econometric Society*, vol. 60, no. 1, pp. 77–105, 1992.

[18] A. Brace, D. Gatarek, and M. Musiela, "The market model of interest rate dynamics," *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, vol. 7, no. 2, pp. 127–147, 1997.

[19] F. Jamshidian, "Libor and swap market models and measures," *Finance and Stochastics*, vol. 1, no. 4, pp. 293–330, 1997.

[20] W. Benner, L. Zyapkov, and S. Jortzik, "A multi-factor cross-currency LIBOR market model," *The Journal of Derivatives*, vol. 16, no. 4, pp. 53–71, 2009.

[21] T.-Y. Hsieh, C.-H. Chou, T.-P. Wu, and C. Son-Nan, "Quanto interest-rate exchange options in a cross-currency libor market model," *Asian Economic and Financial Review*, vol. 5, no. 5, pp. 816–830, 2015.

[22] M. Kijima, K. Tanaka, and T. Wong, "A multi-quality model of interest rates," *Quantitative Finance*, vol. 9, no. 2, pp. 133–145, 2009.

[23] T. Björk, *Arbitrage Theory in Continuous Time*, Oxford University Press, 2009.

[24] IBORate, "Usd libor history," 2018, http://www.iborate.com/gbp-libor/.

[25] IBORate, "Usd libor history," 2018, http://www.iborate.com/usd-libor/.

[26] Fusion Media Limited, "Gbp/usd - british pound us dollar," 2018, https://www.investing.com/currencies/gbp-usd.

[27] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, vol. 23, Springer, New York, NY, USA, 1992.

[28] A. A. Ghaniyyu, P. Ngare, and J. Mung'atu, "Pricing interest rate caps and floors under the Pearson-Sun interest rate model," *Applied Mathematical Sciences*, vol. 12, no. 20, pp. 975–988, 2018.

[29] K. Reddy and V. Clinton, "Simulating stock prices using geometric Brownian motion: Evidence from Australian companies," *Australasian Accounting, Business and Finance Journal*, vol. 10, no. 3, pp. 23–47, 2016.