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Robust Optimal Portfolio and Bank Capital Adequacy Management

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Abstract

The task of jointly estimating an optimal portfolio and bank capital adequacy ratio under model uncertainties is a real and challenging problem to portfolio managers in the banking industry. In this paper, we investigate the problem of optimal portfolio choice of an ambiguity averse portfolio manager (AAPM) with an obligation to continuously meet her bank's capital adequacy requirements as specified in the BASEL III banking agreement. This is carried out through a stochastic modelling approach, requiring model specification uncertainties on three categories of a bank's balance sheet items-assets, capital and liabilities processes driven by independent Brownian motions. We prove the dynamic programming principle and derive the corresponding Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations, leading to a robust capital adequacy ratio and an optimal portfolio selection. We conclude by looking at some numerical applications.

AMS subject classification:
Keywords: Model Uncertainty, Optimal Portfolio, Banking Capital Adequacy, Robust Management.

1. Introduction

Optimal portfolio selection is of fundamental importance in the banking industry for investors maximum utility from terminal wealth. The severity of the 2008 global financial crisis that have been essentially remarkable in the financial market across the world is the catalyst of banking regulations to focus on capital adequacy requirements and ultimately on the choice of an optimal portfolio. In the emerging financial markets, the effects of this crisis is further compounded by low-quality data and illiquidity. If the data quality is low, a portfolio choice would be based on model parameters which are extremely difficult to estimate, creating modelling risk due to ambiguity.

In this paper, we investigate optimal portfolio strategies for investors' maximum utility from terminal wealth under strict conditions on bank capital adequacy requirement and uncertainties in model specifications of economic indicators using stochastic modelling, computational and analytical approaches. The Basel committee on Bank supervision has been established to strengthen the capital regulatory framework. The Basel I contributed to the growth of securisation by assigning lower capital charges to securitised assets, thereby encouraging banks to move assets into off-balance sheet vehicles. Basel II reduced the overall demand for bank capital and, consequently, its cost, leading to lower average rates for both high and low risk firms (see [1]). The Basel III Accord builds on the Basel I and II documents and seeks to improve the banking sector's ability to deal with financial and economic stress, improve risk management, strengthen the banks transparency (see [2]) and increase the amount of money banks hold as capital. Mathematical modelling on portfolio selection dates back to the seminal work of [3]. In mean-variance portfolio selection model, the return on a portfolio is measured by the expected value of the portfolio return, and the associated risk is quantified by the variance of the portfolio return. But as [4] points out, ambiguity aversion seems important in financial markets, where agents are deeply concerned over the level of transparency (i.e. the reliability of the probability distribution of outcomes they refer to).

In this work, we consider the optimal portfolio allocation of a portfolio manager who wants to maximize the utility from terminal wealth of her investors while meeting regulators' capital adequacy requirements for her bank. In addition, she is also concerned about ambiguity in financial market models. We assume that capital allocations in the banking industry and trading in financial market securities happen in continuous time, without taxes or transaction costs. The first mathematical framework for this type of banking problem which is close to our interest, addressing the optimal portfolio selection and capital adequacy management problem by adopting stochastic optimization, Hamilton Jacobi Bellman (HJB) equations, and dynamic programming principles was first proposed in [5].

In a complete market setting, they minimize the capital adequacy risk by solving a nonlinear stochastic optimal control problem by means of the dynamic programming principle with constant interest rate. In [6], an explicit risk aggregation and capital expression was provided regarding the portfolio choice and capital requirements with the Vacisec model as the dynamic of the interest rate in a complete market setting. While control theoretic approaches can be highly useful in optimal portfolio selection and capital adequacy management, they are often constrained by the low quality of real emerging market data. Recently, [7] published a paper on model ambiguity but in the context of insurance investments and portfolio management. In a separate effort to deal with model ambiguity under stochastic volatility, the work of [7] on estimation of volatility memory parameters adapted del Moral's filters - a class of genetic algorithm to finance. In our framework, the balance sheet items for the bank behave unpredictably from uncertain economic activities related to the evolution of treasuries, loan demand, risky and riskless investments, deposits, loans payments, borrowing and eligible regulatory capital. In most economic situations, particularly in emerging financial markets, a portfolio manager cannot be confident of the probability distribution of assets in the market. This can be attributed to model ambiguity in the sense that accurate calibration of model parameters or exact specification of asset and liability models is very difficult to achieve because of illiquidity and partial observation of market data. means that there are more than one equivalent martingale measure in the market. It was suggested in [8], that capital requirement is the main object of consideration in portfolio choice for capital management among practitioners in the banking industry. Through the specification of a capital requirement, a bank identifies what it should invest on and the measures to be taken in order to deal with the benchmark which depends on the bank capital and the total asset portfolio.

This work is a fundamental, transformative shift in portfolio choice and capital adequacy management, enabling accurate modelling of investors' behaviour in the context of emerging financial market characterized by low quality data. This work focuses on understanding the fundamental choices and trade-offs of investments in risky assets in emerging financial markets while ensuring the financial safety and soundness of the banking industry. The effects of model uncertainty, partial observability of key economic parameters on optimal portfolio choice and capital adequacy requirements are investigated. The key mathematical contribution of this paper lie in our proof of the dynamic programming principle. We provide a representation of the dynamic programming principle when the pay-off function is nonlinear. However, we deal with the certainty equivalence in a novel way which relies on the use of worse case probability in quantifying uncertainties. While the nonlinear expectation problem presented in this paper is mathematically interesting in its own right, we emphasize that there is a room for application in robust portfolio selection and estimation of capital adequacy ration in the presence of model ambiguity. For instance, to complement current phenomenological frameworks used to study optimal portfolio selection and capital adequacy management, we examine the optimal choice of an ambiguity averse portfolio manager who wants to maximize the expected utility from terminal wealth of her investors while meeting regulators' capital adequacy requirements.

This paper proceeds as follows: In section 2, we provide our solution to the robust optimization problem arising in the banking industry of a typical emerging financial market. We start with a formal formulation of the problem. In particular, we provide model uncertainty specification for assets, securities and liabilities. The main result is provided through the derivation of a robust optimality principle but the heart of the argument lies mainly in the proof of Proposition 2.2. In section 3, we demonstrate an application to optimal portfolio selection and estimation of the capital adequacy ratio. We consider a model specification uncertainty represented through a vector of independent Brownian motions. We present a numerical simulations which provide some intuition on the relationship between an ambiguity aversion, portfolio strategy and capital adequacy ratio.

2. Main Results

2.1. The Economy and assumptions

In specifying the economy and the banking industry, we consider a continuous-time financial market with the following assumptions: The bank manager can trade continuously in time. Trading and capital reservation in the market involve no extra costs or taxes. We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on a time horizon, T, with filtration, $\{\mathcal{F}_t\},\ t \geq 0$, generated by three independent one-dimensional Brownian motions $\{W_S(t), W_L(t)\},\ t \geq 0$ and $\mathbb{P}$ is a probability measure on Ω.

We also assume that the bank's balance sheet assets, capital and liabilities satisfy the relation,

$$\text{total asset} = \text{total liabilities} + \text{bank capital}.$$

In particular, at any time t, the bank's stylized balance sheet is represented as follows:

$$R(t) + L(t) + S(t) \equiv B(t) + D(t) + C(t), \tag{1}$$

where $R(t)$, $L(t)$, $S(t)$, $B(t)$, $D(t)$ and $C(t)$ are respectively Reserves, Loans, Securities, Deposits, Borrowings and Bank capital. Each of these variables is given as a function defined from $\Omega \times T \longrightarrow \mathbb{R}_+$. We further assume equal proportionality between the reserves and the borrowing plus deposit. Here, the relation in (1) becomes

$$L(t) + S(t) \equiv C(t), \tag{2}$$

Following the standard practice in the banking industry (see [5]), we put the bank securities into two categories; The first contains the risk-less assets called treasuries issued by national treasuries in most countries as means of borrowing money to meet government expenditure not covered by tax revenues. The second group contains risky assets called market securities (e.g. loans, equities etc.). The dynamic of treasury is given as follows:

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt. \tag{3}$$

On the other hand, the dynamic of market security price is given by:

$$\frac{dS(t)}{S(t)} = (r(t) + \lambda_S)dt + \sigma_S dW_S(t) \tag{4}$$

where σ_S is the security volatility and λ_S denotes the risk premium. Under the Capital Asset Pricing Model (CAPM), the risk premium can be quantified by the relation $\lambda_S = \beta[E(R_m) - R_f]$, with $E(R_m)$ representing the market expected return, R_f the risk-free interest rate, and β the sensitivity of the expected excess asset returns to the expected excess market return. The dynamics of the loans is given as follows:

$$\frac{dL(t)}{L(t)} = (r(t) + \lambda_L)dt + \sigma_L dW_L(t), \tag{5}$$

where, $\lambda_L = \lambda_r \sigma_L + \delta$ and δ default risk premium. Then by Itô's formula, the dynamics of the total asset portfolio is represented by the following equation

$$\begin{aligned} \frac{dX(t)}{X(t)} &= (1 - \pi_L(t) - \pi_S(t))\frac{dS_0(t)}{S_0(t)} + \pi_L(t)\frac{dL(t)}{L(t)} + \pi_S(t)\frac{dS(t)}{S(t)}, \\ &= (r(t) + \pi_L(t)\lambda_L + \pi_S(t)\lambda_S)dt + \pi_L(t)\sigma_L dW_L + \pi_S(t)\sigma_S dW_S \\ & \qquad X(0) = X_0. \end{aligned}$$

where $(1 - \pi_L(t) - \pi_S(t))$, $\pi_L(t)$ and $\pi_S(t)$ are the proportions, invested in treasuries, loans and market securities respectively.

In the Basel III accord, the regulatory capital can be divided into Tier 1 and Tier 2 capital (see [9]). Then the bank capital at time $t \geq 0$ is given by;

$$C(t) = C_{T_1}(t) + C_{T_2}(t).$$

The Tier 1 capital which describes the capital adequacy of a bank includes equity capital $E(t)$ and retained earnings. Tiers 2 capital includes subordinated debt $S_D(t)$, limited life preferred stocks, loans losses reserves and good will. Since the nature of retained earnings, life preferred stocks and loan-loss reserves are not dynamic, we do not consider these as active constituents of bank capital. Thus, the total bank capital is given as:

$$C(t) = E(t) + S_D(t).$$

Assuming that the market value of the sum of subordinate debt is given by

$$S_D = S_D(0)e^{\int_0^t r(s)ds}. \tag{6}$$

In what follows, the bank is assumed to hold capital in $n + 1$ categories, one of which is risk free and corresponds to subordinated debt, and n categories for bank equity when each one of them is modelled as follows:

$$\frac{dE(t)}{E(t)} = (r(t) + \lambda_E(t))dt + \sigma_E dW_E(t) \tag{7}$$

where λ_E represents the market price of risk. Hence, the total bank-capital dynamics becomes

$$\begin{aligned} \frac{dC(t)}{C(t)} &= \sum_{i=1}^{n} \pi_i(t)\frac{dE(t)}{E(t)} + \big(1 - \sum_{i=1}^{n} \pi_i(t)\big)\frac{dS_D(t)}{S_D(t)} - \rho X(t)dt \\ &= (r(t) + \pi^{tp}(t)\lambda_E)dt + \pi^{tp}(t)\sigma_E dW_E(t) - \rho X(t)dt \end{aligned}$$

where $\pi^{tp}(t)$ is the transpose vector of the optimal proportions invested in loans and securities. At time t, the bank capital is converted into loans and securities at the rate, $\rho X(t)$ for a constant ρ.

Let $X_r(t)$ denotes the total risk-weighted assets defined by placing each of the on-and-off balance sheet items into a risk category such that the Capital Adequacy Ratio (CAR) is defined as,

$$\text{CAR} = \frac{C(t)}{X_r(t)}. \tag{8}$$

So, one of the portfolio manager's problems is to consider the capital constraints such that CAR $\geq \rho$.

2.2. Robust problem with model ambiguity

A bank manager is allowed to maintain a portfolio of assets and liabilities under strict conditions on bank capital adequacy requirement with uncertainties in economic models of assets, liability and capital reserves. We assume that the manager's aim is to maximize investors' utility from terminal wealth. It is common in the literature to assume that the manager is ambiguity neutral portfolio manager (ANPM) and does not worry about model uncertainties as she has full confidence in the calibrated reference models of the economy and the banking industry. The ANPM maximizes the expected terminal value of the utility function as

$$\max_{\pi \in \bar{\mathcal{A}}} \mathbb{E}_0^{\mathbb{P}} \left[U(X_T) \right], \tag{9}$$

where $\mathbb{E}_t^{\mathbb{P}}[\,\cdot\,] = \mathbb{E}^{\mathbb{P}}[\,\cdot\,|\mathcal{F}_t]$ stands for the conditional expectation under a fixed probability measure, $\mathbb{P}$, and $\bar{\mathcal{A}}$ is the set of admissible strategies (see Definition 2.1).

In most economic situations, particularly in the African financial markets, a bank manager cannot be confident of the probability distribution of assets in the market. This can be attributed to model ambiguity in the sense that accurate calibration of model parameters is very difficult to achieve because of poor reliability of market data. In this paper we incorporate ambiguity by assuming that the portfolio manager does not have full confidence in the reference model and hopes to take other alternative models in consideration. Let $\mathbb{P}$ be a reference probability measure, representing the investor's reference probability model and $\mathcal{Q}$ denote the set of probability measures $\mathbb{Q}$ such that $\mathbb{Q} \sim \mathbb{P}$. Then the robust strategy has to deal with this by following the approach of an equivalent martingale measure $\mathbb{Q}$ to $\mathbb{P}$ (see [10]) and to penalize each such model with a penalty function, $\varphi(\mathbb{Q})$. Now, for a bank with a strictly initial asset value $X(0)$, and share-holders enjoying a power utility function U, the bank manager now faces a problem of maximizing share-holder's utility from terminal wealth with model ambiguity. In order to tackle the ambiguity, the manager has to consider some alternatives to $\mathbb{P}$. Every alternative model is characterized by a stochastic process θ and the associated probability measure $\mathbb{Q}$, which is equivalent to the reference measure $\mathbb{P}$ (see [11]).

The property of the Radon-Nikodym Theorem (see [12]) is used to find the set of equivalent martingale measures to the reference probability measure. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space satisfying the usual conditions. Let $\mathbb{Q}$ be another probability measure on $(\Omega, \mathcal{F})$ under the assumption that $\mathbb{Q} \ll \mathbb{P}$, then there exist a non negative random variable $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z$ and we call Z the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$. Then, by the Cameron-Martin-Girsanov theorem (see [13]), we consider, an adapted process $\theta(t) = (\theta_L(t), \theta_S(t))$ such that for every $\mathbb{Q} \in \mathcal{Q}$, $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z(t)$, where

$$Z(t) = e^{\int_0^t \theta_L(u)dW_L(u) + \int_0^t \theta_S(u)dW_S(u) - \frac{1}{2}\int_0^t (\theta_L^2(u) + \theta_S^2(u))du}$$

and $Z(t)$ is a positive (W_L, W_S) martingale under $\mathbb{P}$ for $0 \leq u < t < T$.

To ensure the martingale property, we assume that $\theta(t)$ satisfies the boundedness condition, that is there exists a constant $c > 0$, $\forall t \in [0, T]$ such that

$$\| \theta(t) \|^2 < c \text{ a.s.} \tag{10}$$

Furthermore, by the Girsanov's theorem to achieve the model uncertainty we allow the drift parameters in the incomplete market to be undetermined. This means that we add the drift term $\theta(t)$ to the independent standard Brownian motions (dW_L, dW_S) under $\mathbb{P}$, and we have

$$\begin{aligned} dW_L^Q &= dW_L + \theta_L(t)dt & (11) \\ dW_S^Q &= dW_S + \theta_S(t)dt & (12) \end{aligned}$$

two standard independent Brownian motion under each $\mathbb{Q} \in \mathcal{Q}$.

A myriad of attributes has been linked to investors in African financial market. Some are ambiguity averse due to complex information structures while others can be regarded as either ambiguity neutral or seeking [14]. We want to determine a unique strategy π^* from the set of admissible strategies $\mathcal{A}$ and a unique measure $\mathbb{Q}^* \in \mathcal{Q}$ such that the portfolio manager can maximize the utility from terminal wealth of her ambiguity aversed investors. From equations (11) and (12) into (2.1), we can write the dynamics of the total asset portfolio under the alternative model $\mathbb{Q}$ as:

$$\begin{aligned} \frac{dX(t)}{X(t)} &= [r(t) + \pi_L(t)(\lambda_L - \sigma_L(t)\theta_L) + \pi_S(t)(\lambda_S - \sigma_S\theta_S(t)]dt \\ &\quad + \pi_L(t)\sigma_L dW_L^{\mathbb{Q}} + \pi_S(t)\sigma_S dW_S^{\mathbb{Q}} \end{aligned} \tag{13}$$

Hence, we can define the set of admissible strategies related to our problem as

Definition 2.1. A strategy, π, from the set;

$$\mathcal{A} = \left\{\pi(\cdot) = \pi(t)_{t\in[0,T]}, \mathcal{F} - \text{adapted.}\right\}$$

is admissible if

- $X_t \geq K$ for a constant $K > 0, 0 \leq t \leq T$;
- $\mathbb{E}^{Q^*}[\int_0^t (\pi_S\sigma_s)^2 + (\pi_L\sigma_L)^2 dt] < \infty$;
- $\forall(t,x) \in [0,T] \times \mathbb{R}$, equation (2.1) has a unique solution $\{X(t)\}_{t\in[0,T]}$ with $\mathbb{E}^{Q^*}[U(X(t)] < \infty$ where Q^* is the model worst-case-scenario probability measure.

This leads to a search for robust optimal strategies of the following form;

$$\sup_{\pi(t)\in\mathcal{A}} \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_t^{\mathbb{Q}(\pi)}\big[U(X_T) + \varphi(\mathbb{Q}(\pi))\big] \tag{14}$$

where, $\mathcal{A}$ is the set of control processes for an ambiguity-neutral regulator in a given market and $E_t^{\mathbb{Q}}[\,\cdot\,] = E^{\mathbb{Q}}[\,\cdot\,|F_t]$ represents the conditional expectation under the probability measure $\mathbb{Q}$.

2.3. Solution via a worse case scenario probability

In our work, we propose that the worse case scenario measure $Q^*(\pi)$ is the required unique measure $Q^*(\pi) \in \mathcal{Q}$ that satisfies the above conditions. To determine this measure explicitly and the corresponding value function $V(t,x)$, we set up and solve a stochastic optimal control problem through the dynamic programming approach in a Brownian motion setting. In particular, the following expression holds

$$V(t,x) = \mathbb{E}_t^{Q^*(\pi^*)}[U(X_T)] = \sup_{\pi\in\mathcal{A}} \inf_{\mathbb{Q}\in\mathcal{Q}} \Big\{\mathbb{E}_t^{\mathbb{Q}(\pi)}\big[U(X_T) + \varphi(\mathbb{Q}(\pi))\big]\Big\} \tag{15}$$

We now solve the corresponding Hamilton-Jacobi-Bellman-Isaacs (HJBI) [15] equation for a power utility function over the time horizon T. That is:

$$U(X) = \frac{X^{1-\gamma}}{1-\gamma},\ 0 < \gamma < 1. \tag{16}$$

and the boundary condition given by $V(T,x) = U(x)$. This implies that

$$\sup_{\pi\in\mathcal{A}} \inf_{(\theta_L,\theta_S)\in\theta(t)} \{\triangle^{\pi} V + \varphi(\theta(t))\} = 0 \tag{17}$$

where the differential operator $\triangle^{\pi}$ is computed as follows;

$$\begin{aligned}\triangle^{\pi} V = V_t + V_x x\big[r(t) + \pi_L(t)(\lambda_L - \sigma_L(t)\theta_L(t)) + \pi_S(t)(\lambda_S - \sigma_S\theta_S(t)\big] \\ + \frac{1}{2} V_{xx} x^2\big[(\pi_L(t)\sigma_L)^2 + (\pi_S(t)\sigma_S)^2\big]\end{aligned} \tag{18}$$

Specifically, since any $\mathbb{Q} \in \mathcal{Q}$ is defined with respect to the pair $(\theta_L(t), \theta_S(t))$ we define a map Q^* taking any strategy π to $(\theta_L(t), \theta_S(t))$. Then the penalty function will be determined by a map taking any of the strategy satisfying (10). That is

$$\varphi(\theta(t)) = \frac{1}{2\phi}\left(\theta_L^2(t) + \theta_S^2(t)\right) \tag{19}$$

Where ϕ is the preference parameter which governs the ambiguity aversion.

By [16], we impose that the preference parameter $\phi > 0$ depends on the state variable (the current wealth process $V(t,x)$) in order to ensure that the penalty function is reasonable. Then we choose the preference parameter ϕ to be given by $\phi = \dfrac{\alpha}{(1-\gamma)V(t,x)} > 0$. Where $\alpha > 0$ indicates the portfolio manager's ambiguity aversion level.

Proposition 2.2. The solution of (17) is given by $V(t,x) = f(t)\dfrac{x^{1-\gamma}}{1-\gamma}$. Where the boundary conditions $f(T) = 1$, $f(t) = e^{\beta(T-t)}$ with $\beta = (1-\gamma)\left[r + \dfrac{\left(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right)}{2\left(\alpha+\gamma\right)}\right]$ and the optimal investment strategy given by

$$\pi_L^*(t) = \frac{\lambda_L}{\sigma_L^2\left(\alpha+\gamma\right)} \tag{20}$$

$$\pi_S^*(t) = \frac{\lambda_S}{\sigma_S^2\left(\alpha+\gamma\right)} \tag{21}$$

Proof. Assuming that $V(t,x) = f(t)\dfrac{x^{1-\gamma}}{1-\gamma}$, then we have

$$V_t = f'(t)\frac{x^{1-\gamma}}{1-\gamma} \tag{22}$$

$$V_x = f(t)x^{-\gamma} \tag{23}$$

$$V_{xx} = -\gamma f(t)x^{-1-\gamma} \tag{24}$$

For the derivation of the worst-case drifts, we differentiate (17) with respect to θ_L, and θ_S, that is maximizing over $\mathcal{Q}$. Then we have

$$-\sigma_L x V_x \pi_L(t) + \frac{1}{\phi}\theta_L(t) = 0$$

$$-\sigma_S x V_x \pi_S(t) + \frac{1}{\phi}\theta_S(t) = 0.$$

This implies that

$$\theta_L^*(t) = \sigma_L \phi x V_x \pi_L(t) \tag{25}$$

$$\theta_S^*(t) = \sigma_S \phi x V_x \pi_S(t) \tag{26}$$

Replacing (25) and (26) into (17) gives

$$\sup_{\pi\in\mathcal{A}}\Big\{V_t + V_x x\big[r + \pi_L(t)\big(\lambda_L - \sigma_L^2 x V_x \phi \pi_L(t)\big) + \pi_S(t)\big(\lambda_S - \sigma_S^2 x V_x \phi \pi_S(t)\big)\big] + \frac{1}{2}V_{xx}x^2\big[(\pi_L(t)\sigma_L)^2 + (\pi_S(t)\sigma_S)^2\big] + \frac{\phi}{2}\big[\big(\sigma_L^2 x V_x \pi_L(t)\big)^2 + \big(\sigma_S^2 x V_x \pi_S(t)\big)^2\big]\Big\} = 0 \quad (27)$$

For the derivation of the optimal robust investment strategy, differentiating (27) with respect to $\pi_L(t)$ and $\pi_S(t)$ implies that

$$x V_x \lambda_L + \sigma_L^2 x^2\big(V_{xx} - V_x^2\phi\big)\pi_L(t) = 0$$
$$x V_x \lambda_S + \sigma_S^2 x^2\big(V_{xx} - V_x^2\phi\big)\pi_S(t) = 0$$

Then we have the optimal investment strategy given by

$$\pi_L^*(t) = \frac{V_x\lambda_L}{x\sigma_L^2\big(V_x^2\phi - V_{xx}\big)} \quad (28)$$

$$\pi_S^*(t) = \frac{V_x\lambda_S}{x\sigma_S^2\big(V_x^2\phi - V_{xx}\big)} \quad (29)$$

Now, substituting (28) and (29) into (27) yields

$$V_t + xV_x\big[r + \frac{V_x\lambda_L^2}{\sigma_L^2 x(V_x^2\phi - V_x x)}\big(1 - \frac{\phi V_x^2}{(V_x^2\phi - V_x x)}\big) + \frac{V_x\lambda_S^2}{\sigma_S^2 x(V_x^2\phi - V_x x)}\big(1 - \frac{\phi V_x^2}{(V_x^2\phi - V_x x)}\big)\big] + \frac{1}{2}V_{xx}\big[\big(\frac{V_x\lambda_L}{\sigma_L(V_x^2\phi - V_{xx})}\big)^2 + \big(\frac{V_x\lambda_S}{\sigma_S(V_x^2\phi - V_{xx})}\big)^2\big] + \frac{\phi}{2}\big[\big(\frac{V_x^2\lambda_L}{\sigma_L(V_x^2\phi - V_x x)}\big)^2 + \big(\frac{V_x^2\lambda_S}{\sigma_S(V_x^2\phi - V_x x)}\big)^2\big] = 0$$

Which implies that

$$V_t + xV_x\big[r + \frac{V_x}{x(V_x^2\phi - V_{xx})}\big(1 - \frac{\phi V_x^2}{(V_x^2\phi - V_{xx})}\big)\big(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\big)\big] + \frac{1}{2}V_{xx}\big(\frac{V_x}{(V_x^2\phi - V_{xx})}\big)^2\big(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\big) + \frac{\phi}{2}\big(\frac{V_x^2}{(V_x^2\phi - V_{xx})}\big)^2\big(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\big) = 0$$

$$\Rightarrow V_t + xV_x\Big[r + A(1-B)\Big(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\Big)\Big] + \frac{1}{2}V_{xx}C^2\Big(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\Big) + \frac{\phi}{2}D^2\Big(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\Big) = 0$$

$$\Rightarrow V_t + rxV_x + \Big(A(1-B)xV_x + \frac{1}{2}C^2V_{xx} + \frac{\phi}{2}D^2\Big)\Big(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\Big) = 0$$

Where

$$\begin{aligned}
A &= \frac{V_x}{x(V_x^2\phi - V_{xx})} \\
&= \frac{f(t)x^{-\gamma}}{x\left(f^2(t)x^{-2\gamma}\phi + \gamma f(t)x^{-1-\gamma}\right)} \\
&= \frac{f(t)x^{-\gamma}}{x\left(f^2(t)x^{-2\gamma}\frac{\alpha}{f}(t)x^{1-\gamma} + \gamma f(t)x^{-1-\gamma}\right)} \\
&= \frac{1}{\alpha+\gamma}
\end{aligned}$$

$$\begin{aligned}
B &= \frac{\phi V_x^2}{(V_x^2\phi - V_{xx})} \\
&= \frac{\alpha f^2(t)x^{-2\gamma}}{f^2(t)x^{-2\gamma}\alpha + \gamma f^2(t)x^{-2\gamma}} \\
&= \frac{\alpha}{\alpha+\gamma}
\end{aligned}$$

Where

$$\begin{aligned}
C &= \frac{V_x}{(V_x^2\phi - V_{xx})} \\
&= \frac{f(t)x^{-\gamma}}{\frac{f^2(t)x^{-2\gamma}\alpha}{\frac{(1-\gamma)f(t)x^{1-\gamma}}{1-\gamma}} + \gamma f(t)x^{-1-\gamma}} \\
&= \frac{x}{\alpha+\gamma}
\end{aligned}$$

$$\begin{aligned}
D &= \frac{V_x^2}{(V_x^2\phi - V_{xx})} \\
&= \frac{f^2(t)x^{-2\gamma}}{\frac{\alpha f^2(t)x^{-2\gamma}}{\frac{(1-\gamma)f(t)x^{1-\gamma}}{1-\gamma}} + \gamma f(t)x^{-1-\gamma}} \\
&= \frac{f(t)x^{1-\gamma}}{\alpha+\gamma}
\end{aligned}$$

Replacing A, B, C and D by their values, we then have

$$
\begin{aligned}
&V_t + rxV_x + \Big(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\Big)\Big[\frac{1}{\alpha+\gamma}\Big(1 - \frac{\alpha}{\alpha+\gamma}\Big)xV_x \\
&+\frac{1}{2}\Big(\frac{x}{\alpha+\gamma}\Big)^2 V_{xx} + \frac{\phi}{2}\Big(\frac{f(t)x^{1-\gamma}}{\alpha+\gamma}\Big)^2\Big] = 0 \\
&\Longrightarrow V_t + rxV_x + \Big(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\Big)\Big[\frac{\gamma}{\big(\alpha+\gamma\big)^2}xV_x \\
&+\frac{1}{2}\Big(\frac{x}{\alpha+\gamma}\Big)^2 V_{xx} + \frac{\phi}{2}\Big(\frac{f(t)x^{1-\gamma}}{\alpha+\gamma}\Big)^2\Big] = 0 \\
&\Longrightarrow V_t + rxV_x + \Big(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\Big)\frac{1}{\big(\alpha+\gamma\big)^2} \\
&\Big[\gamma xV_x + \frac{1}{2}x^2V_{xx} + \frac{\phi}{2}\Big(f(t)x^{1-\gamma}\Big)^2\Big] = 0
\end{aligned}
$$

Substituting (22), (23) and (24) we have the following equation

$$
\begin{aligned}
&f'(t)\frac{x^{1-\gamma}}{1-\gamma} + rf(t)x^{1-\gamma} + \Big(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\Big)\frac{1}{\big(\alpha+\gamma\big)^2} \\
&\Big[\gamma f(t)x^{1-\gamma} - \frac{\gamma}{2}f(t)x^{1-\gamma} + \frac{\alpha}{2}f(t)x^{1-\gamma}\Big] = 0 \\
&\Longrightarrow \frac{f'(t)}{1-\gamma} + rf(t) + \Big(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\Big)\frac{1}{\big(\alpha+\gamma\big)^2} \\
&\Big(\gamma - \frac{\gamma}{2} + \frac{\alpha}{2}\Big)f(t) = 0 \\
&\Longrightarrow \frac{f'(t)}{1-\gamma} + \Big[r + \Big(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\Big)\frac{1}{2\big(\alpha+\gamma\big)}\Big]f(t) = 0 \\
&\Longrightarrow f'(t) + (1-\gamma)\Big[r + \frac{\big(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\big)}{2\big(\alpha+\gamma\big)}\Big]f(t) = 0
\end{aligned}
$$

Therefore we have

$$f'(t) + \beta f(t) = 0. \tag{30}$$

Where

$$\beta = (1-\gamma)\Big[r + \frac{\big(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\big)}{2\big(\alpha+\gamma\big)}\Big].$$

By equation (30) we have $f(t) = e^{\beta(T-t)}$ and $f(T) = 1$. Therefore $V(t,x) = e^{\beta(T-t)}\dfrac{x^{1-\gamma}}{1-\gamma}$ and $V(T,x) = \dfrac{x^{1-\gamma}}{1-\gamma}$ with the optimal investment strategy given by (20) and (21). ■

From proposition 2.2, the optimal investment strategy is the classical optimal solution for [17] with the risk-aversion adjustment replaced by $\alpha + \gamma$.

3. Numerical Application

3.1. Calibration and interpretation of the ambiguity aversion level α

In order to predict the quantitative effect of robustness on the portfolio choice given by (20) and (21), we suggest the calibration of the parameter α.

From (20) and (21) we have

$$\begin{aligned} \theta_L^*(t) &= \sigma_L \phi x V_x \pi_L(t) \\ &= \frac{\alpha \lambda_L}{\sigma_L(\alpha+\gamma)} \end{aligned} \tag{31}$$

$$\begin{aligned} \theta_S^*(t) &= \sigma_S \phi x V_x \pi_L(t) \\ &= \frac{\alpha \lambda_S}{\sigma_S(\alpha+\gamma)} \end{aligned} \tag{32}$$

Then table 1 reports the optimal investments strategies allocated to the total assets and the worst-case drifts from various values of the ambiguity level α.

Setting the risk premium $\lambda_L = 6\%$, the loan volatility $\sigma_L = 16\%$, $\lambda_S = 5\%$ and $\sigma_S = 18\%$.

Table 1: Optimal portfolio shares and worst-case drifts

	$\gamma = 0.1$					$\gamma = 0.5$				
α	π_L	π_S	$1-\pi_L-\pi_S$	θ_L	θ_S	π_L	π_S	$1-\pi_L-\pi_S$	θ_L	θ_S
0	23.4375	15.4321	−37.8696	0	0	4.6875	3.0864	−6.7739	0	0
0.1	11.7188	7.7160	−18.4348	0.1875	0.1389	3.9063	2.5720	−5.4783	0.0625	0.0463
0.4	4.6875	3.0864	−6.7739	0.3000	0.2222	2.6042	1.7147	−3.3188	0.1667	0.1235
0.5	3.9063	2.5720	−5.4783	0.3125	0.2315	2.3438	1.5432	−2.887	0.1875	0.1389
0.7	2.9297	1.9290	−3.8587	0.3281	0.2430	1.9531	1.2860	−2.2391	0.2187	0.1620
0.8	2.6042	1.7147	−3.3188	0.3333	0.2469	1.8029	1.1871	−1.9901	0.2308	0.1709
1	2.1307	1.4029	−2.5336	0.3409	0.2525	1.5625	1.0289	−1.5914	0.2500	0.1852

The observation of table 1 and figure 1 shows that the preference for robustness decreases the optimal portfolio weight because looking at the values of π_L and π_S we see that the more the ambiguity aversion level α increases, the more the the optimal portfolio weight decreases. But unlikely for the worst-case drifts we see that the parameters θ_L and θ_S increases with the ambiguity aversion level. If the parameter $\alpha = 0$, then the

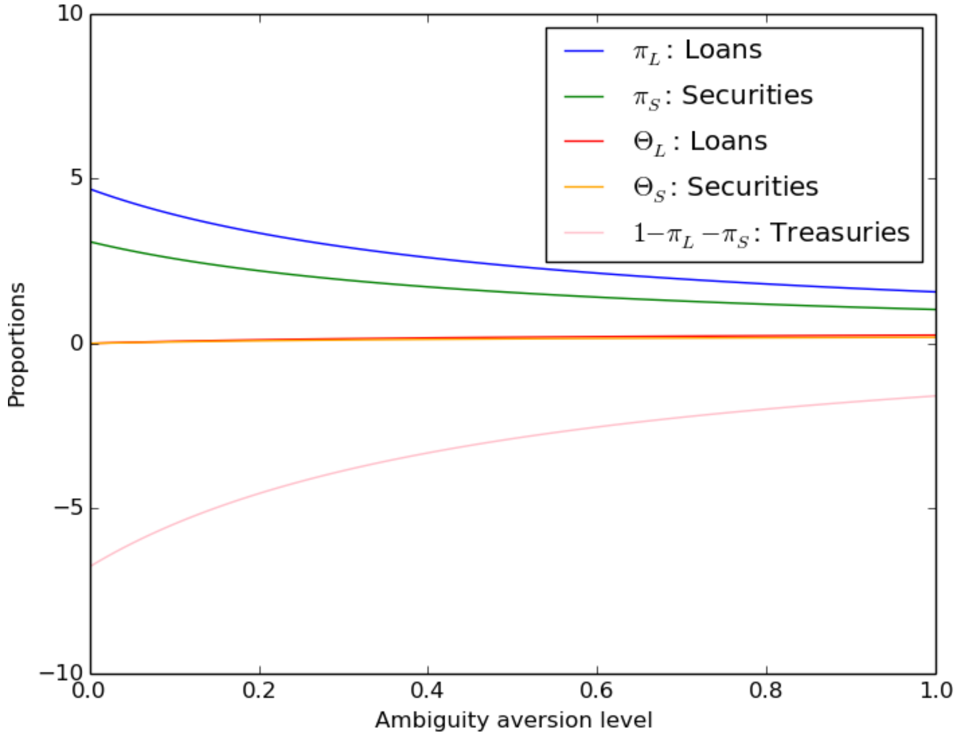

Figure 1: Optimal Investment Portfolio Weights and worst-case drifts

worst-case drifts vanishes and the Ambiguity Averse Portfolio reduces to the Ambiguity Neutral.

By figure 1 for the risk-aversion $\gamma = 0.5$ and we can be able to see the behaviour of the optimal investment weights and the worst case drifts in response to the different values of the ambiguity averse level α and we see that the curves of the drifts remain constants as α tends to infinity, but since the penalty term will vanish at that time, the Ambiguity Averse Portfolio manager has to look for another alternative model.

Now that we have been able to find out the optimal values of the parameters for the wealth process, in the following section we use them to find the capital adequacy.

3.2. Bank Capital Adequacy Management

By the Basel III accord the risk weighted asset is given as

$$\begin{aligned}\frac{dX_r(t)}{X_r(t)} &= \Big[0.5\pi_L(t)\big(r(t)+\lambda_L-\sigma_L\theta_L(t)\big)+0.2\pi_S(t)\\ &\big(r(t)+\lambda_S-\sigma_S\theta_S(t)\big)\Big]dt+0.5\sigma_L\pi_L(t)dW_L^Q+0.2\sigma_S\pi_S(t)dW_S^Q\end{aligned} \tag{33}$$

By (8) and Itô formula, the derivation of the capital adequacy ratio will be given as

$$
\begin{aligned}
d\frac{C(t)}{X_r(t)} &= \frac{dC(t)}{X_r(t)} - \frac{C(t)dX_r(t)}{X_r^2(t)} + \frac{C(t)(dX_r(t))^2}{X_r^3(t)} \\
&\quad - \frac{dX_r(t)dC(t)}{X_r^2(t)} \\
&= \frac{C(t)}{X_r(t)}\big[(r(t)+\pi^{tp}(t)\lambda_E)dt + \pi^{tp}(t)\sigma_E dW_E(t)\big] \\
&\quad -\rho\frac{X(t)}{X_r(t)}dt - \frac{C(t)}{X_r(t)}\big[0.5\pi_L(t)(r(t) \\
&\quad +\lambda_L - \sigma_L\theta_L(t)) + 0.2\pi_S(t)(r(t)+\lambda_S \\
&\quad -\sigma_S\theta_S(t))\big]dt + \frac{C(t)}{X_r(t)}\big[(0.5\pi_L\sigma_L)^2 \\
&\quad +(0.2\pi_L\sigma_L)^2\big]dt - \frac{C(t)}{X_r(t)}\big[0.5\pi_L(t)\sigma_L dW_L^Q \\
&\quad +0.2\pi_S(t)\sigma_S dW_S^Q\big] \\
&= \big[C_r(t)(a_1-b_1)-c_1\big]dt \\
&\quad -C_r(t)\big[a_2 dW_L^Q + b_2 dW_S(t) - c_2 dW_E(t)\big]
\end{aligned}
$$

Where

$$
\begin{aligned}
a_1 &= r(t)+\pi^{tp}(t)\lambda_E \\
b_1 &= 0.5\pi_L(t)\big(r(t)+\lambda_L-\sigma_L\theta_L(t)\big) \\
&\quad +0.2\pi_S(t)\big(r(t)+\lambda_S-\sigma_S\theta_S(t)\big) - (0.5\sigma_L\pi_L(t))^2 \\
&\quad -(0.2\sigma_S\pi_S(t))^2 \\
c_1 &= \rho\frac{X(t)}{X_r(t)} \\
a_2 &= 0.5\sigma_L\pi_L(t) \\
b_2 &= 0.2\sigma_S\pi_S(t) \\
c_2 &= \pi^{tp}(t)\sigma_E
\end{aligned}
$$

and $C_r(t)$ denotes the bank capital adequacy ratio. Compared with the dynamic of the capital adequacy ratio in [6], the ambiguity parameter has only impact on the drift term.

Then this shows the behaviour of the dynamic of the Capital Adequacy Ratio in figure 2.

With the initial value fixed at 48.94%, the capital adequacy ratio moves up and down over time and has a value of just over 96.11%.

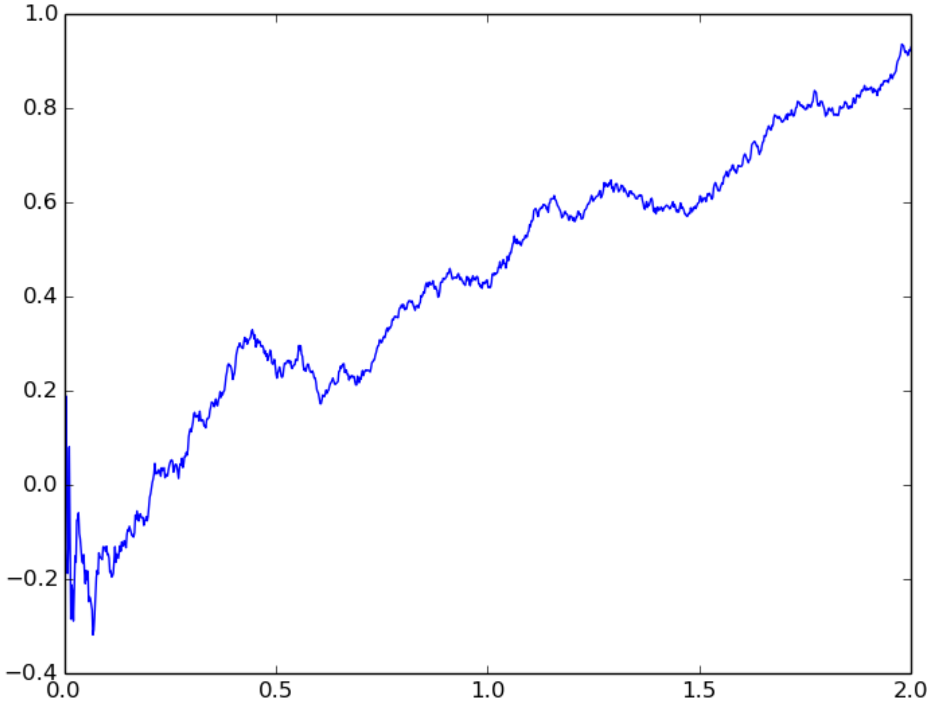

Figure 2: Dynamic of the Capital Adequacy Ratio(CAR)

4. Conclusion

The aim of this work was to find the optimal investment strategy under strict capital adequacy requirement for an ambiguity averse manager where the term structure of interest rate is constant. Our contribution was to obtain an optimal investment allocation strategy that optimizes the bank's asset portfolio consisting of balance sheet items.

This was achieved by constructing the stochastic differential equations satisfied by the total asset portfolio in the presence of model specification uncertainty on three categories of a bank's balance sheet items and developing the investment strategy that maximizes the bank portfolio by means of dynamic programming with a power utility function.

Furthermore, we derived the dynamic of the capital adequacy ratio by determining the dynamic of the risk weighted assets under Basel III agreement. Our results show that for an ambiguity averse portfolio manager, the value of the capital adequacy ratio maintains the minimum required. This result is great helpful in determining the optimal investment allocation strategy and the corresponding adequate capital in a financial market where the uncertainty arises.

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