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Full Length Research Paper

Estimation of T- period's ahead extreme quantile autoregression function

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This paper considers the estimation of extreme quantile autoregression function by using a parametric model. We combine direct estimation of quantiles in the middle region with that of extreme parts using the model and results from extreme value theory (EVT). The volatility used to scale the residuals is estimated indirectly, without estimating conditional mean, using the conditional quantile (CQ) range. The estimators are found to be consistent. A small simulation study carried out shows that the estimator of the volatility function converges to the true function over a range of distributional errors. Finally, the T-periods ahead extreme quantile autoregression function is given.

Key words: Quantile, autoregression, value-at-risk, ARCH, extreme value theory, consistency, asymptotic normality.

INTRODUCTION

Many financial institutions have switched from risk management based on accrual accounting to management based on daily marking to market. This has seen an increase in the volatility of the apparent value of overall positions held by financial institutions, which now reflects the volatility of the underlying markets and effectiveness of hedging strategies. Theoretically, one can measure financial risks by using measures such as standard deviation, quantile, interquantile range or expected shortfall. The quantile based Value-at-Risk (VaR) has become a basic tool employed by financial institutions and their regulators to assess riskiness of trading activities. Based on negative returns, VaR is defined as quantile of returns at high or extreme probability level. Specifically, VaR is defined so that the probability that a portfolio will loss more than its quantile over a particular time horizon is equal to φ, for $\varphi \to 1$ specified. Because of the conceptual simplicity, quantile models have been adopted for regularity purposes. In particular, the 1996 market risk amendments, which allow ten-day 1% VaR to be measured as a multiple of one day, have been adopted for regulatory purposes. The Basel Accord (1996) stipulates that Banks and Broker dealer's minimum capital requirements for market risk should be set based on the ten-day 1% VaR for the trading portfolios. Detail analysis and application of the quantiles to risk measurement can be found in among others, Morgan (1996), Duffie and Pan (1997), Jorion (1997), Dowd (1998) and stulz (1998). Although the quantile risk measures do not satisfy the sub-additivity condition of a good risk measure, like expected shortfall, the later depends on the accuracy of the quantile estimator for its accuracy, see Mwita (2003), McNeil and Frey (2000), among others.

The extreme quantiles correspond to extreme probability levels. These quantiles can be estimated by using results from EVT, Embrechts et al. (1997), for surveys of Mathematical Theory of EVT and applications to both financial and insurance risk management. EVT can be used to characterise the behaviour of extreme returns or the returns distribution without tying the analysis down to a single parametric family fitted to a whole distribution. However, because of the presence of volatility in the financial data, it is inappropriate to apply the results directly. Furthermore, Danielson and de Vries (1998) have shown that these results do not work well in probabilities as low as 0.95. Few attempts have been made

Abbreviations: VaR, Value-at-risk; **EVT,** extreme value theory; **CQ,** conditional quantile; **QAR,** quantile autoregressive; **QARCH,** quantile autoregressive conditional heteroscedastic.

to develop extensions of extreme value statistical methodology to take account of the volatility. Among others, McNeil and Frey (2000) and Barone-Adesi et al. (1998) have taken an approach built around the Generalized Autoregressive Conditional Heteroscedastic (GARCH) with heavy tailed innovation estimated using EVT results. Mwita (2003) proposed a class of nonparametric time series models, called quantile autoregressive - quantile autoregressive conditional heteroscedastic (QAR-QARCH) and used it to estimate conditional extreme quantiles. In that work, the QAR function at relatively high probability level, say θ, constituted a threshold beyond which scaled iid residuals are assumed to follow heavy tailed distribution. The QARCH function was used to scale and transform the dependent excesses to iid excesses beyond the threshold. The asymptotic properties of the nonparametric estimators of the QAR and QARCH functions can be found in Mwita (2003) and Mwita and Otiena (2005). In this paper, we follow similar concepts as in Mwita (2003), but instead of assuming the conditional scale function to be in the form of quantile autoregressive function, we assume the conditional scale is the conditional standard deviation. However, we do not make assumptions on the finiteness of the first moment, as in the case of Engle (1982). The volatility was estimated using the interquantile autoregressive function.

THE MODEL AND ESTIMATORS

Let $\{Y_t, \underset{\sim}{X}_t\} \in \mathbf{R} \times \mathbf{R}^P$ such that $Y_t \in \mathrm{F}_t$ and $\underset{\sim}{X}_t \in \mathrm{F}_{t-1}$ are measurable. A good example of such time series is the autoregressive series, where Y_t depends on its own past $Y_{t-1}, Y_{t-2},,,,,$. We will assume that $\{Y_t, \underset{\sim}{X}_t\}$ are from a conditional distribution $F_{\underset{\sim}{X}_t}(Y_t)$ that is at point $(y, \underset{\sim}{x})$, $F(y \mid \underset{\sim}{x}) = F_{\underset{\sim}{x}}(y) = P(Y_t \le y \mid \underset{\sim}{X}_t = \underset{\sim}{x})$. Let $\theta \in (0,1)$. Then conditional θ^{th} quantile function of Y_t given $\underset{\sim}{X}_t$, called quantile autoregression function (QARF), is defined as the inverse function, that is

$$\mu(\underset{\sim}{x}, \beta_\theta) = \inf\{y \in \mathbf{R} \mid F(y \mid \underset{\sim}{x}) \ge \theta\}$$

Where $F(y \mid \underset{\sim}{x})$ is assumed to have a continuous conditional density $f(y \mid \underset{\sim}{x})$. We introduce and define a quantile autoregression-autoregression conditional heteroscedastic (QAR-ARCH) process as;

$$Y_t = \mu(\underset{\sim}{X}_t, \beta_\theta) + \sigma(\underset{\sim}{X}_t, \alpha) Z_t \tag{1}$$

Where;

$\mu(\underset{\sim}{X}_t, \beta_\theta)$ Is a θ^{th}-quantile autoregression (QAR) function of Y_t given $\underset{\sim}{X}_t$. $\sigma(\underset{\sim}{X}_t, \alpha)$ Is a volatility or ARCH function of Y_t given $\underset{\sim}{X}_t$. Z_t are the iid random variables with zero θ-quantile and a unit scale. β_θ and α are vectors of parameters defined in (3).

In particular, Z_t are iid with zero quantile and unit standard deviation. A specific form of model (1) is an AR-ARCH process of the form

$$Y_t = \mu(\underset{\sim}{X}_t, \beta) + \sigma(\underset{\sim}{X}_t, \alpha) e_t \tag{2}$$

Where $\mu(\underset{\sim}{X}_t, \beta)$ is the conditional mean (e.g. AR part) and e_t is iid with mean zero and standard deviation of 1. Simple manipulation of (2) gives $Z_t = e_t - q_\theta^e$ with the above properties, where q_θ^e is the quantile of $\{e_t\}$. In (2), it is assumed that $\{e_t\}$ is symmetric and $q_\theta^e = 0$. Model (1) generalizes (2) because no assumptions are made on the first moment and the type of distribution. Using similar notations as in Koenker and Basset (1978) and Engle (1982), we define parametrically a QAR (p)-ARCH (q) process as

$$\begin{aligned} Y_t &= \beta_{0,\theta} + \beta_{1,\theta} Y_{t-1} + \beta_{2,\theta} Y_{t-2} + \dots + \beta_{P,\theta} Y_{t-P} + \sqrt{\alpha_0 + \alpha_1 Y_{t-1}^2 + \alpha_2 Y_{t-2}^2 + \dots + \alpha_q Y_{t-q}^2}\, Z_t \\ &= \beta_\theta' \underset{\sim}{X}_t + \sqrt{\alpha' \underset{\sim}{X}_t^2}\, Z_t \end{aligned} \tag{3}$$

Where:

$\beta_\theta = (\beta_{0,\theta}, \beta_{1,\theta}, \beta_{2,\theta} \dots \beta_{P,\theta})', \underset{\sim}{X}_t = (1, Y_{t-1}, Y_{t-2}, \dots, Y_{t-P})', \alpha = (\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_q)'$ and $\underset{\sim}{X}_t^2 = (1, Y_{t-1}^2, Y_{t-2}^2, \dots, Y_{t-q}^2)'$. Stationarity condition for QAR (p) in (3) is given as

$$\left| \sum_{i=1}^{P} \beta_{i,\theta} \right| < 1 \tag{4}$$

Note that since we are using interquantile autoregression range to determine the standard deviation, then the stationarity condition will also depend on (4). The parametric vector β_θ is the solution to the following minimization problem, assuming integrability and the uniqueness of the solution,

$$\beta_\theta = \underbrace{\arg\min}_{\beta} E\left[\rho_\theta(Y_t - \beta' \underset{\sim}{X}_t)\right] \tag*{5)}$$

Where, for $x \in \mathbf{R}$, $\rho_\theta(x) = x\left(\theta - I_{\{x \le 0\}}\right)$, is called the check function. The sample version of β_θ, based on a random sample $\{(X_1,Y_1),(X_2,Y_2),...,(X_n,Y_n)\}$ of $\{\underset{\sim}{X}_t,Y_t\}$ is given by,

$$\hat{\beta}_\theta = \underbrace{\arg\min}_{\beta}\frac{1}{n}\sum_{t=1}^{n}\rho_\theta(Y_t-\beta'\underset{\sim}{X}_t) \qquad (6)$$

The estimator for $\mu(\underset{\sim}{X}_t,\beta_\theta)$ then becomes

$$\mu(\underset{\sim}{X}_t,\hat{\beta}_\theta)=\hat{\beta}_\theta'\underset{\sim}{X}_t \qquad (7)$$

We define the Interquantile Autoregressive Range at θ denoted by $IQAR_\theta$ as $IQAR_\theta=\mu(\underset{\sim}{X}_t,\beta_\theta)-\mu(\underset{\sim}{X}_t,\beta_{1-\theta})$. Using model (2), we have the conditional standard deviation or volatility in this case as $\sigma(X_t,\alpha)=\dfrac{IQAR_\theta}{q_\theta^e-q_{1-\theta}^e}$ and corresponding estimator of the range is $I\hat{Q}AR_\theta=\mu(\underset{\sim}{X}_t,\hat{\beta}_\theta)-\mu(\underset{\sim}{X}_t,\hat{\beta}_{1-\theta})$. Once the distribution of $\{e_t\}$ is approximated, q_θ^e and $q_{1-\theta}^e$ can be obtained. Note that although the volatility is specified as dependent on α, the method of interquantile autoregression range only depends β_θ. The consistency and asymptotic normally of $\hat{\sigma}(X_t,\alpha)$ will follow that of $\mu(\underset{\sim}{X}_t,\hat{\beta}_\theta)$. We therefore only give the asymptotic properties of $\mu(X_t,\hat{\beta}_\theta)$.

Asymptotic properties of the estimator, $\mu(X_t,\hat{\beta}_\theta)$

Consider again process (1). The following assumptions are important to guarantee the consistency of $\mu(X_t,\hat{\beta}_\theta)$.

(A1) (F,F,P) is a complete probability space and $\{Z_t,X_t\},t=1,2,...,$ are random vectors on this space.

(A2) The function $\mu(\underset{\sim}{X}_t,\beta_\theta):\mathbf{R}^{k_t}\times B\to\mathbf{R}$ is such that for each β_θ in B, a compact subset of $\Re^p$, $\mu(\underset{\sim}{X}_t,\beta_\theta)$ is measurable with respect to the Borel set B^P and $\mu(\underset{\sim}{X}_t,\cdot)$ is continuous in B, a.s- $P,t=1,2,...$ for a given choice of explanatory variables $\{X_t\}$.

(A3) (a) $E([\rho_\theta(Y_t-\beta_\theta'\underset{\sim}{X}_t)])$ exists and is finite for each β_θ in B.
(b) $E([\rho_\theta(Y_t-\beta_\theta'\underset{\sim}{X}_t)])$ is continuous in β_θ.
(c) $\{[\rho_\theta(Y_t-\beta_\theta'\underset{\sim}{X}_t)]\}$ obeys the strong (weak) law of large numbers.

For example, we could assume the $\{Z_t,X_t\}$ are α-mixing. That is $\alpha(m)$ satisfies $\alpha(n)\to 0$ as $n\to\infty$. (Andrews, 1988; White and Domowitz, 1984).

(A4) $\{n^{-1}E\{[\rho_\theta(Y_t-\beta_\theta'\underset{\sim}{X}_t)]\}\}$ has identifiably unique maximizers.

Theorem 1 (Consistency)

In process (1), under assumptions (A1)-(A4), $\hat{\beta}_\theta\to\beta_\theta$ as $n\to\infty, a.s-P_0$, where $\hat{\beta}_\theta$ is given by (6). The proof is similar to the one in White (1994 p. 75), by using the check function defined above. To prove the asymptotic normality of $\hat{\beta}_\theta$, we introduce some extra notation. Let v_t be a $(r\times 1)$ vector of variables that determine the shape of the conditional distribution of $\varepsilon_t=\sigma_t Z_t$. Associated with v_t is a set of parameters ϕ. Denote the density of ε_t, conditional on all the past information, as $h_t(\varepsilon,\phi,v_t)$, $\varepsilon\in\Re$. Here, v_t includes conditional variance and ϕ, the vector of parameters that define a volatility model. Whenever the dependence on v_t and ϕ is not relevant, we will denote the conditional density of ε_t simply by $h_t(\varepsilon)$. Let $u_t(\phi,\beta_\theta,s)$ be an unconditional density of $s_t=(\varepsilon_t,X_t,v_t)$. Finally, define the operators $\nabla\equiv\partial/\partial\beta$, $\nabla_i\equiv\partial/\partial\beta_i$, where β_i is the ith element of β, and $\nabla_i\mu_t(\beta)\equiv\nabla_i\mu(\underset{\sim}{X}_t,\beta)$ and $\nabla\mu_t(\beta)\equiv\nabla\mu(\underset{\sim}{X}_t,\beta)$. The following assumptions are important for asymptotic normality.

(B1). $\nabla_i\mu_t(\beta_\theta)$ is A-smooth with variables A_{it} and functions $\kappa_i, i=1,2,...,p$. In addition, $\max_i\kappa_i(d)\le d$ for small enough.

(B2) (i) $h_t(\varepsilon)$ is Lipschitz continuous in ε uniformly in t.
(ii) For each t and $(\varepsilon,v),h_t(\varepsilon;\phi,v)$ is continuous in ϕ.

(B3) For each t $s, u_t(\phi, \beta_\theta, s)$ is continuous in (ϕ, β_θ).

(B4) $\{\varepsilon_t, X_t\}$ are α-mixing, with parameter $\alpha(n)$, and there exist $\Delta < \infty$ and $r > 2$ such that $\alpha(n) \leq \Delta n^{\omega}$ for some $\omega < -2r/(r-2)$.

(B5) For some $r > 2, \nabla_i \mu_t(\beta)$ is uniformly r-dominated by functions a_{1t}.

(B6) For all t and i, $E\,|\sup_\beta A_{it}|^r\ \Delta_1 < \infty$. There exist a measurable functions a_{2t} such that $|u_t| \leq a_{2t}$ and for all t, $\int a_{2t} dv < \infty$ and $\int a_{1t}{}^3 a_{2t} dv < \infty$.

(B7) There exists a matrix A such that $n^{-1} \sum_{t=a+1}^{a+n} E[\nabla \mu_t(\beta_\theta) \nabla' \mu_t(\beta_\theta)] \to A$ as $n \to \infty$, uniformly in a.

Theorem 2 (Asymptotic normality)

In process (1), if (B1)-(B7) hold and if the estimator is consistent, then;

$$\sqrt{\frac{n}{\theta(1-\theta)}} A_n^{-\frac{1}{2}} D_n(\hat{\beta}_\theta - \beta_\theta) \xrightarrow{d} N(0,1)$$

Where:

$$A_n = n^{-1} \sum E[\nabla \mu_t(\beta_\theta) \nabla' \mu_t(\beta_\theta)], D_n = n^{-1} \sum E[h_t(0) \nabla \mu_t(\beta_\theta) \nabla' \mu_t(\beta_\theta)]$$

and $\hat{\beta}_\theta$ is given by (6) and Theorem 1.

Proof

The proof is obtained by substituting $sign(x) = 2[1/2 - I_{\{x \leq 0\}}]$ with the function, $[\theta - I_{\{x \leq 0\}}]$ in theorem 3 of Weiss (1991).

Note that the continuity of $\mu(\underset{\sim}{X}_t, \beta)$ in β can be approximated arbitrarily well by continuous and differentiable functions. Since taking these approximations does not affect the nature of the model, we can treat it as if it satisfies all the necessary assumptions to give consistency and asymptotic normality results.

Extreme quantile and t-periods ahead extreme quantile autoregression functions

Application of quantiles to measure financial risks is normally done in high levels of probability or beyond the maximum observation. The quantiles, which are located among the largest observations or even beyond the data maximum, are called extreme quantiles (Mwita, 2003). To estimate extreme quantiles, results from extreme value theory (Embrecht et al., 1997) may be used. We take the QAR at relatively high probability level, say θ, as an initial as well as the beginning of the right-hand tail of a heavy tailed distribution. This is then combined with quantiles obtained using Gnedenko's result (Gnedenk, 1943) and a Hill's estimator (Hill, 1975) of the tail index to arrive at an approximate extreme QAR function at high probability levels, say $\varphi > \theta$.

The limiting distributions of the sample maxima are given by the following theorem due to Fisher and Tippet (1928).

Theorem 3

Suppose $e_1, e_2, ..., e_n$ is a sequence of iid random variables from unknown distribution F and $M_n = \max(e_1, e_2, ..., e_n)$ denotes the maximum of the first n observations. If a sequence of real numbers $a_n > 0$ and $b_n \in \mathbf{R}$ can be found such that the sequence of normalized maxima, $\frac{M_n - b_n}{a_n}$, converges in distribution for some non-degenerate distribution function H, then H belongs to one of the three distribution types:

$$H_\xi(e) = \begin{cases} \exp\left\{-(1+\xi e)^{-\frac{1}{\xi}}\right\}, \xi \neq 0 \\ \exp\{-\exp(-e)\}, \xi = 0 \end{cases}$$

Where e is such that $1 + \xi e > 0$, and ξ is the shape parameter. A random variable e_t is said to belong to the Maximum Domain of Attraction (MDA) of the extreme value distribution H if and only if the Fisher-Tippet theorem hold for $\{e_t\}$.

Consider the random variable $\{e_t\}$ in (1) and (2). To derive an estimate of an extreme quantile q_φ^e for $\varphi \approx 1$, we take a relatively high quantile of $\{e_t\}$ at θ, denoted as $q_\theta{}^e$, where $\theta < \varphi$ is large but not so close to 1 as φ.

The probability φ is so large that we have none or only a few data in our sample beyond q_φ^e. The quantile q_θ^e is assumed to be the threshold above which a Pareto like tail holds. In other words q_φ^e is the quantile of $e_t = e_\theta + z_t$. The following proposition gives the extreme quantile, q_φ^e.

Proposition 1

Let the iid variable $\{e_t\}$ be in the maximum domain of attraction of $H_\xi, \xi > 0$. Suppose θ is a high probability corresponding to the quantile q_θ^e above which a Pareto like tail holds, then the extreme conditional quantile of e_t is given as

$$q_\varphi^e \approx \left(\frac{1-\varphi}{1-\theta}\right)^{-\xi} q_\theta^e \text{, for large } \theta \text{ and } \varphi > \theta.$$

Proof

The quantiles q_θ^e and q_φ^e correspond, respectively, to the excess probabilities $\overline{F}(q_\theta^e) = 1-\theta$ and $\overline{F}(q_\varphi^e) = 1-\varphi$. Then using Gnedenko's result, $\overline{F}(e) = e^{-\frac{1}{\xi}} L(e), e > 0$, the excess probabilities satisfy

$$\overline{F}(q_\theta^e) = (q_\theta^e)^{-\frac{1}{\xi}} L(q_\theta^e) \tag{8}$$

and

$$\overline{F}(e) = (e)^{-\frac{1}{\xi}} L(e), e > q_\theta^e \tag{9}$$

Dividing (9) by (8) and noting that for large θ, $L(e)/L(q_\theta^e) \approx 1$, we obtain

$$e \approx \left(\frac{1-F(e)}{1-\theta}\right)^{-\xi} q_\theta^e \text{, for large } \theta. \tag{10}$$

Setting $F(e) = \varphi > \theta$, the φ-quantile is obtained as the inverse;

$$q_\varphi^e \approx \left(\frac{1-\varphi}{1-\theta}\right)^{-\xi} q_\theta^e \text{, for large } \theta \text{ and } \varphi > \theta. \tag{11}$$

The estimator of (11) is

$\hat{q}_\varphi^e \approx \left(\frac{1-\varphi}{1-\theta}\right)^{-\hat{\xi}} q_\theta^e$, assuming the distribution of e_t is a known heavy tailed.

Note that $\hat{\xi}$ is the Hill's estimator of the shape parameter, obtained as

$$\hat{\xi} = \frac{1}{N_\theta} \sum_{t=1}^{n} \log\left(\frac{e_t}{q_\theta^e}\right) I_{\{e_t > q_\theta^e\}} \tag{12}$$

With N_θ being the number of exceedances. The obtained estimator was consistent (Hill, 1975). Proposition 2 extends the estimation of extreme quantiles in the iid case to dependent case, by augmenting the QAR with Gnedenko's result and the Hill's estimator of the shape parameter in (12).

Proposition 2

Assume the random variable $Y_t, t = 1, 2, ...$ in process (1) and the iid error $\{e_t\}$ with $F \in MDA(H_\xi, \xi > 0)$. The extreme QAR, at θ, is given by

$$\mu_\varphi(\underset{\sim}{X}_t, \beta_\theta) = \mu_\theta(\underset{\sim}{X}_t, \beta_\theta) + \sigma(\underset{\sim}{X}_t, \alpha) q_\theta^e \left(\left(\frac{1-\varphi}{1-\theta}\right)^{-\xi} - 1\right)$$

Proof of proposition 2

From (1), we have $\frac{Y_t - \mu(\underset{\sim}{X}_t, \beta_\theta)}{\sigma(\underset{\sim}{X}_t, \alpha)} = Z_t = e_t - q_\theta^e$, with $\mu_\theta(\underset{\sim}{X}_t, \beta_\theta)$ being the threshold above which the scaled variable Z_t is assumed to follow a GPD. Now, $\{e_t\}$ is iid and q_θ^e is the threshold. The quantile of the excess variable Z_t is

$$\frac{\mu_\varphi(\underset{\sim}{X}_t, \beta_\theta) - \mu(\underset{\sim}{X}_t, \beta_\theta)}{\sigma(\underset{\sim}{X}_t, \alpha)} = q_\varphi^z = q_\varphi^e - q_\theta^e \text{, with } \varphi > \theta$$

Using equation (11), we get

$$\frac{\mu_{\varphi}(\underset{\sim}{X}_t,\beta_\theta)-\mu(\underset{\sim}{X}_t,\beta_\theta)}{\sigma(\underset{\sim}{X}_t,\alpha)} \approx \left(\frac{1-\varphi}{1-\theta}\right)^{-\xi} q_\theta^e - q_\theta^e$$

and

$$\mu_{\varphi}(\underset{\sim}{X}_t,\beta_\theta)=\mu(\underset{\sim}{X}_t,\beta_\theta)+\sigma(\underset{\sim}{X}_t,\alpha)q_\theta^e\left(\left(\frac{1-\varphi}{1-\theta}\right)^{-\xi}-1\right).$$

(13)

The estimator of $\mu_{\varphi}(\underset{\sim}{X}_t,\beta_\theta)$ is clearly given as

$$\hat{\mu}_{\varphi}(\underset{\sim}{X}_t,\hat{\beta}_\theta)=\mu_\theta(\underset{\sim}{X}_t,\hat{\beta}_\theta)+\hat{\sigma}(\underset{\sim}{X}_t,\alpha)q_\theta^e\left(\left(\frac{1-\varphi}{1-\theta}\right)^{-\hat{\xi}}-1\right)$$

(14)

The method of obtaining a T-step period prediction can be based on the work in Feller (1971, VIII.8), where it is shown that the tail risk for fat tailed distributions is, to a first order approximation, linearly additive. Therefore, assuming q_θ^e is the threshold above which a Pareto like tail holds, and using (10), the one period prediction based on iid random variable $\{Z_t\}$ is

$$\frac{\mu_{\varphi}(\underset{\sim}{X}_{t+1},\beta_\theta)-\mu(\underset{\sim}{X}_{t+1},\beta_\theta)}{\sigma(\underset{\sim}{X}_{t+1},\alpha)} \approx q_\varphi^z$$

The T-period ahead based on unconditional random variable $Z_t = e_t - q_\theta^e$ is clearly seen as

$$\frac{\mu_{\varphi}^T(\underset{\sim}{X}_{t+1},\beta_\theta)-\mu(\underset{\sim}{X}_{t+1},\beta_\theta)}{\sigma(\underset{\sim}{X}_{t+1},\alpha)} \approx T^{\xi} q_\varphi^z$$

Rearranging, we obtain an estimator of T-period ahead extreme quantile autoregression function as

$$\hat{\mu}_{\varphi}^T(\underset{\sim}{X}_{t+1},\hat{\beta}_\theta) \approx \mu_\theta(\underset{\sim}{X}_{t+1},\hat{\beta}_\theta)+T^{\xi}\hat{\sigma}(\underset{\sim}{X}_{t+1},\alpha)q_\varphi^z,$$

(15)

Where the components $\mu(\underset{\sim}{X}_{t+1},\hat{\beta}_\theta), \hat{\sigma}(\underset{\sim}{X}_{t+1},\alpha)$ and $\hat{\xi}$ are consistent estimators given above.

SIMULATION STUDY

To further reinforce the mathematical results obtained, we used numerical method to investigate properties of the volatility estimator using model (1). We simulated samples of sizes 500, 800 and 1000 from a specific AR (1)-ARCH (1) process

$$Y_t = 0.00022 + 0.9Y_{t-1} + \sqrt{0.07+0.9Y_{t-1}^2}\,e_t,$$

(16)

Under different distribution of the error e_t, that is Standard Normal, Student - t (4), Cauchy and Gamma (2, 2). For all the samples, we estimated the interquantile autoregression range (IQAR) function when $\theta = 0.1$ and 0.9. The computed volatility, $\hat{\sigma}(X_t,\alpha)$, was compared with the true volatility.

To illustrate, take for example one sample with standard normal errors, but without replication. The points in Figure 1 indicates the AR (1) - ARCH (1) process, whereas the dotted lines are the QAR function estimates at 0.9 and 0.1. The points in Figure 2 depict the true volatility and the superimposed dotted line is the estimated. The structure of the estimated volatility function seems to follow same pattern as the true one. Figure 3 shows the AR (1) - ARCH (1) process (points) with the superimposed QAR (dotted line) at $\varphi = 0.99$ obtained using (14). The threshold was taken as $\theta = 0.9$. Using all replications, the properties/performance of the estimator of the volatility was assessed by its Average Mean Absolute Proportionate Error (AMAPE),

$$AMAPE(\hat{\sigma}(X_t,\alpha)) = \frac{1}{100}\sum_{j=1}^{n_j}\left[\frac{1}{n_j}\sum_{i=1}^{n_j}\left|\frac{\hat{\sigma}^{(j)}(X_i,\alpha)-\sigma^{(j)}(X_i,\alpha)}{\sigma^{(j)}(X_i,\alpha)}\right|\right], n_j = 500, 800, 1000$$

The results are shown in Table 1. The AMAPE decreases with the increase in sample size. This confirms the theoretical results on convergence obtained in section 3. It is therefore expected that as $n \to \infty, \hat{\sigma}(X_t,\alpha) \to \sigma(X_t,\alpha)$.

Conclusion

We have introduced a class of parametric models, called QAR-ARCH, and used it to estimate the volatility under specified distribution of the data, and extreme quantile autoregression function. Both mathematical and numerical simulations have confirmed that the estimator of the volatility obtained using the interquantile autoregression range is consistent. The simulation study indicates that further investigation could be carried out to determine, among others, rate of convergence of the volatility estimator.

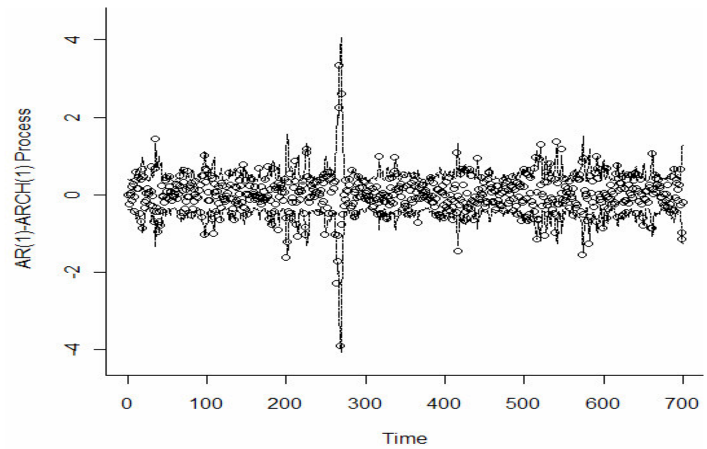

Figure 1. AR (1)-ARCH (1) process (points) with estimated QAR functions at $\theta = 0.1$ and 0.9 (dotted line) superimposed.

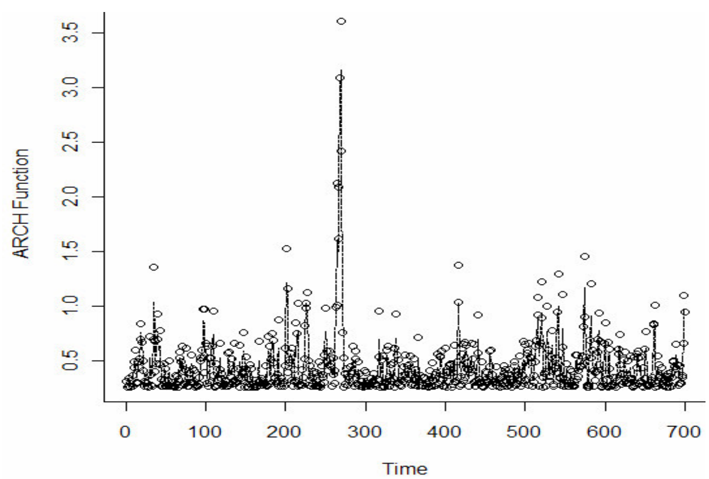

Figure 2. True volatility (points) with estimated volatility (dotted lines) superimposed.

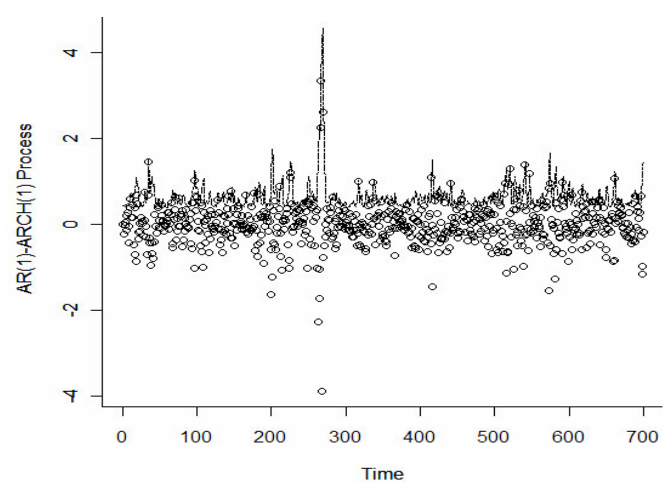

Figure 3. AR (1)-ARCH (1) process (points) with estimated QAR function at $\varphi = 0.99$ (dotted line) superimposed.

Table 1. Average mean absolute proportionate error with increasing sample sizes under different distribution errors.

Error \ Sample size	500	800	1000
Normal (0,1)	0.247	0.228	0.148
Student-t(4)	0.298	0.245	0.180
Cauchy	0.732	0.690	0.440
Gamma	0.645	0.634	0.473

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