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Multiplicativity of the Generating Functions for Refined Node Polynomials on Toric Surfaces

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Contents

Introduction

The advent of tropical enumerative geometry [**Mik05, BM07, FM10**], led to emergence of combinatorial strategies for tackling enumerative problems on complex algebraic varieties. In this thesis, we use *long-edge graphs*, which are combinatorial abstractions of tropical plane curves to prove that the generating function for the refined node polynomials on a subclass of toric surfaces exhibits a multiplicative structure. Long edge graphs were originally introduced by Block, Colley and Kennedy [**BCK14**] and Liu [**Liu16**] to study similar questions for the standard Severi degrees.

In Chapter 1, we set out by briefly establishing the notations and basic definitions that would be used in the thesis. Thereafter, we discuss the theory of refined curve counting on complex surfaces. The main goals of this chapter is threefold. First, we introduce the *refined invariants* (Definition 1.25), the *refined Severi degrees* (Definition 1.29) and the *Welschinger invariants* (Definition 1.37). The refined invariants are Laurent polynomials in a variable y and polynomials in the intersection numbers of the pair (S, L). The refined Severi degrees, defined for toric surfaces and in particular $\mathbb{P}^2, \Sigma_m$ and $\mathbb{P}(1, 1, m)$, are also given by polynomials (called refined node polynomials), for sufficiently ample line bundles. These are also Laurent polynomials in a variable y and polynomials in the intersection numbers of the pair (S, L). The refined Severi degrees are defined in such a way that they specialize at $y = 1$ to the usual Severi degrees and at $y = -1$ to the tropical Welschinger invariants. Second, we state the conjecture (Conjecture 1.28) which asserts that the generating function for the refined invariants has a multiplicative structure. Finally, we state the conjectural relationship between the refined Severi degrees and the refined invariants. This in particular leads to a conjecture that the generating function for the refined node polynomials as well as the generating function for the tropical Welschinger invariants has a multiplicative structure.

In Chapter 2, we begin by a quick introduction to tropical curves. This introduction is not exhaustive, we cherry pick only the fundamental aspects necessary for the purposes of the thesis. Next is an exposition revolving around a correspondence theorem by Mikhalkin (Theorem 2.23). Roughly speaking, Mikhalkin's correspondence theorem [**Mik05**], asserts that weighted counts of tropical curves passing trough sufficiently many tropical point configurations on $\mathbb{R}^2$ is equi-numerous to the counts of complex algebraic curves in $(\mathbb{C}^*)^2$. One of the goals here is to introduce a non-recursive definition of the refined Severi degrees. The underlying theme however, is a description of a pathway on which one can start from counting tropical curves in $\mathbb{R}^2$ to refined tropical curve curve counting. By abstracting the combinatorial properties of tropical curves, we are led to a combinatorial strategy of counting algebraic curves on complex surfaces. The methods used here does not work for all surfaces in general, they apply only to toric surfaces defined by *h-transverse* lattice polygons (Definition 2.28).

We prove the main results of this thesis in Chapter 3. We start by discussing the fundamental aspects of *long-edge graphs* (Definition 3.4), that would be necessary for achieving our results. Associated to each long edge graph G is its *refined multiplicity* and its *cogenus* (Definition 3.5). The first step in achieving the results is proving Theorem 3.18, asserting that the refined Severi degree $N^{(S,L)\delta}(y)$ is equal to the weighted count (with refined multiplicity as weight) of long edge graphs of cogenus δ. The refined Severi degrees are given by *refined node polynomials* which are polynomials in the intersection numbers $LK_S, L^2, K_S^2, \chi(\mathcal{O}_S)$ of the pair (S, L) [**BG16**, Thm. 4.2]. To say that the generating function of the refined node polynomials is multiplicative in the intersection numbers $LK_S, L^2, K_S^2, \chi(\mathcal{O}_S)$ of the pair (S, L) is equivalent to saying that the coefficient of t^δ in the formal logarithm

$$\log\ \mathcal{N}(S, L; y) = \sum_{\delta=1}^{\infty} Q_\delta(S, L)(y)t^\delta$$

of the generating function of the refined node polynomials, is a $\mathbb{Q}[y^{\pm1}]$-linear combination of $LK_S, L^2, K_S^2, \chi(\mathcal{O}_S)$. This equivalent statement is what we prove in Theorem 3.22. In this theorem, we have only considered the case where (S, L) is $(\Sigma_m, cF + dH)$, $(\mathbb{P}^2, dH)$ or $(\mathbb{P}(1, 1, m), dH)$. This also covers the case of $(\mathbb{P}^1 \times \mathbb{P}^1, cF + dH)$. We remark that with more care, one may achieve similar results

(Theorem 3.18 and Theorem 3.22) for general toric surfaces associated to general h-transverse lattice polygons.

In §3.2 we couple Theorem 3.22 with computer calculations to provide more evidence for a conjecture by Göttsche and Shende [**GS14**, Conj 62]. The conjecture says that the generating function for the refined invariants is also multiplicative in the intersection numbers $LK_S, L^2, K_S^2, \chi(\mathcal{O}_S)$ of the pair (S, L). In Corollary 3.29, we extend the bounds for their conjecture in the particular cases of $\mathbb{P}^2$ and Σ_m. The refined Severi degree $N^{(S,L),\delta}$ specialize at $y = -1$ to the tropical Welschinger numbers. This leads to a statement of a conjecture analogous to the conjecture by Göttsche and Shende [**GS14**, Conj 62] for the Welschinger numbers. In Corollary 3.31, we give evidence with much higher bounds for this analogous conjecture.

In §3.3 and §3.4 we study an emergent conjectural principle which is described as follows. for the pair (S, L) of a projective surface and a line bundle, write

$$A^{(S,L)}(y,q) := \frac{B_1(y,q)^{K_S^2} B_2(y,q)^{LK_S} D\widetilde{DG}_2(y,q)}{\left(\widetilde{\Delta}(y,q) D\widetilde{DG}_2(y,q)\right)^{\chi(\mathcal{O}_S)/2}}.$$

To each condition c that is imposed at points of S to curves in $|L|$ on S e.g. passing through points with a given multiplicity, S having singularity at some points there corresponds a power series $F_c \in \mathbb{Q}[y^{\pm 1}][\![q]\!]$ satisfying the following. For L sufficiently ample, the refined count of curves in $|L|$ satisfying conditions $c_1, \ldots, c_n$ is

$$\operatorname*{Coeff}_{q^{(L^2 - LK_S)/2}} \left[\prod_{i=1}^{n} F_{c_i}(y,q) A^{(S,L)}(y,q) \right].$$

In §3.3 we study this principle for the case of toric surface with singularities. Here, we require that the curves counted do not pass through the singular points of the surface. In Proposition 3.38 we provide evidence for this conjectural principle (formulated formally in Conjecture 3.36 and Conjecture 3.37) for the particular case of $\mathbb{P}(1,1,m)$ with a singularity of type $\frac{1}{m}(1,1)$, i.e. the cyclic quotient of $\mathbb{C}^2$ by the m-th roots of unity μ_m acting by $\epsilon(x,y) = (\epsilon x, \epsilon y)$. What's remarkable is that for an A_n singularity, the *correction power series* $F_c \in \mathbb{Q}[y^{\pm 1}][\![q]\!]$ is independent of the variable y. Thus we get the same correction terms for the standard Severi degrees and the Welschinger numbers. In particular, the correction terms coincide with the correction terms for the Severi degrees studied by Liu and Osserman [**LO14**, Thm. 1.8]. Thus the conjectural principle includes the theorem of Liu and Osserman as a special case. We state the specialized conjectures in Conjecture 3.39, Conjecture 3.42 and Conjecture 3.45 and give evidence in Proposition 3.46 for the case of $\mathbb{P}(1,1,2)$.

Finally, in §3.4 we study the conjectural principle in the case of counting curves with multiple points on smooth surfaces. To each point p_i on S on which we impose a condition that curves pass with multiplicity n_i corresponds a correction term $H_i \in \mathbb{Q}[y^{\pm 1}][\![q]\!]$ satisfying the conjectural principle described above. We give the formal statement of the conjecture in Conjecture 3.48. The particular case of $\mathbb{P}^2$ and Σ_1 is stated in Conjecture 3.50 and give evidence in Proposition 3.51.

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CHAPTER 1

Refined Curve Counting on Surfaces

1.1. Notations and Basic Definitions

1.1.1. Ampleness. We begin by recalling a few fundamental notions which will be used in one way or another in the sequel. Let X be a noetherian scheme over $\mathbb{C}$ and L a line bundle on X. We recall that L is said to be globally generated if there exist sections $s_0, \ldots, s_n \in H^0(X, L)$ such that for any $x \in X$ the germs $(s_i)_x$ of the sections s_i at x generate the stalk L_x as a module. This amounts to L having finitely many sections such that for any $x \in X$ there is at least one section not vanishing there. Such a choice of generators defines a morphism

$$\begin{aligned} \varphi : X &\to \mathbb{P}^n \\ x &\mapsto [s_0(x) : \ldots : s_n(x)] \end{aligned} \tag{1.1}$$

such that $L = \varphi^*(\mathcal{O}(1))$. One can therefore say that L is globally generated if there exists a morphism $\varphi : X \to \mathbb{P}^n$ such that $L = \varphi^*(\mathcal{O}(1))$. L is said to be very ample (relative to $\mathrm{Spec}(\mathbb{C})$) if there is a closed immersion $i : X \to \mathbb{P}^n$ such that $L \cong i^*(\mathcal{O}(1))$. Thus in particular a globally generated line bundle L is very ample if the corresponding morphism is an immersion. If X is projective over $\mathbb{C}$, $\mathcal{O}(1)$ is a very ample line bundle on X and $\mathcal{F}$ a coherent sheaf, then by a theorem of Serre [**Har77**, II.5.11] we have that $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}(1)^{\otimes n}$, for some n large enough, is generated by a finite number of global sections. This property is used in the definition of a more general notion of an *ample* line bundle.

Definition 1.1. A line bundle L on a noetherian scheme X is said to be ample if for every coherent sheaf $\mathcal{F}$ on X, there exists an integer n_0 depending on $\mathcal{F}$ such that for every $n \geq n_0$, $\mathcal{F} \otimes L^{\otimes n}$ is generated by global sections.

The notion of an ample line bundle is more general than the notion of a very ample line bundle and in many ways is more convenient to to work with. Another related notion is that of a *k-very ample* line bundle for k an integer. We recall that a 0-dimensional cycle Z of length k on a projective scheme over $\mathbb{C}$ is a purely 0-dimensional subscheme $Z \subset X$ such that $\dim H^0(\mathcal{O}_Z) = k$.

Definition 1.2. A line bundle L on a projective scheme X over $\mathbb{C}$ is said to be k-very ample if for an integer k the restriction map

$$H^0(X, L) \to H^0(L \otimes \mathcal{O}_Z) \tag{1.2}$$

is surjective for every 0-dimensional cycle $Z \subset X$ of length less or equal to $k+1$.

As a remark, the notion of a 0-*very ample* line bundle corresponds to that of an globally generated line bundle whereas being 1-*very ample* corresponds to being *very ample*. It is worth pointing out the following (though we will not use it in the sequel). If L is k-very ample then the map (1.2) associates to every zero dimension cycle $Z \subset X$ of length $k+1$ to a subspace of $H^0(X, L)$ of codimension $k+1$ and this map yields a morphism

$$\varphi_k : X^{[k+1]} \to \mathrm{Gr}(k+1, H^0(X, L))$$

where $\mathrm{Gr}(k+1, H^0(X, L))$ denotes the Grassmanian of all 1-dimensional quotients of $k+1$-dimensional quotients of $H^0(X, L)$ sending $(Z, \mathcal{O}_Z)$ to the quotient $H^0(X, L) \to H^0(L \otimes \mathcal{O}_Z)$. Catanese and Gottsche [**CG90**], proved that if S is a smooth connected surface then a line bundle L on S is k-very ample if and only if the morphism $\varphi_j : S^{[j+1]} \to \mathrm{Gr}(j+1, H^0(S, L))$ is an embedding for every $j \leq k$

1.1.2. Linear Systems of Divisors. Throughout this thesis, we shall be interested mainly in line bundles on surfaces. Let S be a smooth projective surface over $\mathbb{C}$ and let L be a line bundle on S. Assume that $h^0(S, L) > 0$ i.e. there exists a non-zero section $s \in H^0(S, L)$. Then $D = (s)_0$ is an effective Cartier divisor (divisor of zeros) such that $L \cong \mathcal{O}_S(D)$ [**Har77**, II Prop. 7.7]. Furthermore, two sections s_1, s_1 have the same divisor if and only if $s_1 = \lambda s_2$ for $\lambda \in \mathbb{C}^*$.

Definition 1.3. Let S and L be as above and assume that $L \cong \mathcal{O}_S(D)$ for D a Cartier divisor on S. We denote by $|L|$ the *complete linear system* of all effective (Cartier) divisors linearly equivalent to D (also denoted by $|D|$).

As S is smooth over $\mathbb{C}$ (therefore normal) the notion of Cartier divisors on S is equivalent to the notion of Weil divisors on S [**Har77**, II §6]. Thus the elements of

$|L|$ are finite formal linear combinations (with non-negative integer coefficients) of closed codimension one subvarieties of S. By a curve on S in the linear system $|L|$ we mean a reduced but possibly reducible effective divisor $C \in |L|$. A curve C is said to be irreducible if it is not a sum of two non-trivial effective divisors. There exists a bijective correspondence between $|L|$ and the set $\mathbb{P}H^0(S,L) = (H^0(S,L) - \{0\})/\mathbb{C}^*$ and this gives the linear system $|L|$ a structure of the set of closed points of a projective space over $\mathbb{C}$ [**Har77**, II Prop. 7.7].

1.1.3. Intersection Product on $\mathrm{Pic}(S)$**.** Let S be a smooth projective surface over $\mathbb{C}$. We denote by $\mathrm{Div}(S)$ the group of all divisors on S and $\mathrm{Pic}(S)$ the group of invertible sheaves (line bundles) up to isomorphism. $\mathrm{Pic}(S)$ is isomorphic to the group of divisors modulo linear equivalence (called divisor class group and denoted Cl S) [**Har77**, Cor. II.6.16]. We also have that $\mathrm{Pic}(S) \cong H^1(S, \mathcal{O}_S^*)$. To see this consider the exact sequence of sheaves

$$1 \to \mathcal{O}_S^* \to \mathscr{K}^* \to \mathscr{K}^*/\mathcal{O}_S^* \to 1 \tag{1.3}$$

where $\mathscr{K}$ is the sheaf of total quotient rings of $\mathcal{O}_S$ (since S is integral, $\mathscr{K}$ is just the constant sheaf corresponding to the function field K of S). Taking the sheaf cohomology and using the fact that $H^0(S, \mathcal{O}_S^*) = \mathbb{C}^*$, $H^0(S, \mathscr{K}^*) = \mathscr{K}^*$ and $H^0(S, \mathscr{K}^*/\mathcal{O}_S^*) = \mathrm{Div}(S)$ we get the long exact sequence

$$0 \to \mathbb{C}^* \to \mathscr{K}^* \xrightarrow{pr} \mathrm{Div}(S) \to H^1(S, \mathcal{O}_S^*) \to H^1(S, \mathscr{K}^*) \to \cdots \tag{1.4}$$

where the map $pr : \mathscr{K}^* \to \mathrm{Div}(S)$ associates to a non-zero rational function $f \in \mathscr{K}^*$ the corresponding principal divisor $div(f)$. $\mathscr{K}$ is constant and therefore flasque which implies that $H^1(S, \mathscr{K}^*) = 0$. Consequently $\mathrm{Pic}(S) := \mathrm{Div}(S)/pr(\mathscr{K}^*) \cong H^1(S, \mathcal{O}_S^*)$.

An important feature of the Picard group in the case of surfaces is due to the existence of an intersection form which we now describe. Let D_1 and D_2 be two distinct curves (effective divisors) on S having no irreducible component and $x \in D_1 \cap D_2$ be a point. The intersection multiplicity of D_1 and D_2 at x is defined to be $m_x(D_1 \cap D_2) := \dim_{\mathbb{C}} \mathcal{O}_{S,x}/(f,g)$ where f, g are local equations of D_1 and D_2 respectively [**Bea96**, Def. I.2]. The intersection number of D_1 and D_2 is then defined to be

$$D_1 \cdot D_2 := \sum_{x \in D_1 \cap D_2} m_x(D_1 \cap D_2)$$

This definition of intersection number is symmetric, bilinear and depends only on the linear equivalence classes of D_1 and D_2 [**Har77**, V, Prop 1.4]. Therefore it extends to a symmetric bilinear pairing $\mathrm{Pic}(S) \times \mathrm{Pic}(S) \to \mathbb{Z}$. In fact, let $L_1, L_2 \in \mathrm{Pic}(S)$ then the pairing

$$L_1 \cdot L_2 := \chi(\mathcal{O}_S) - \chi(L_1^*) - \chi(L_2^*) + \chi(L_1^* \otimes L_2^*),$$

where L^* denotes the dual sheaf to L and $\chi(L) = \sum_i (-1)^i h^i(S, L)$ is the Euler characteristic, defines a symmetric bilinear pairing on $\mathrm{Pic}(S)$ such that if D_1 and D_2 are distinct irreducible divisors on S then [**Bea96**, Thm. 1.4]

$$\mathcal{O}_S(D_1) \cdot \mathcal{O}_S(D_2) = D_1 \cdot D_2.$$

The intersection pairing defined (algebraically) above can also be defined using topologically. Here we use the fact that the category of schemes over $\mathbb{C}$ is isomorphic to the category of complex analytic spaces (GAGA [**Ser56**]). Consider the exact sequence of analytic sheaves (exponential sequence) on S.

$$0 \to \mathbb{Z} \to {}^h\mathcal{O}_S \to {}^h\mathcal{O}_S^* \to 1$$

where ${}^h\mathcal{O}_S$ denote the sheaf of holomorphic functions on S (Considered as an analytic manifold). In the associated long exact sequence in cohomology, we have the following

$$\cdots \to H^1(S, \mathbb{Z}) \to H^1(S, {}^h\mathcal{O}_S) \to H^1(S, {}^h\mathcal{O}_S^*) \xrightarrow{c_1} H^2(S, \mathbb{Z}) \to \cdots$$

Thus by the GAGA correspondence $\mathrm{Pic}(S) \cong H^1(S, \mathcal{O}_S^*) \cong H^1(S, {}^h\mathcal{O}_S^*)$. We therefore have a map $c_1 : \mathrm{Pic}(S) \to H^2(S, \mathbb{Z})$ mapping a line bundle L to its image $c_1(L)$ called the *Chern class* of L.

The map $c_1 : \mathrm{Pic}(S) \to H^2(S, \mathbb{Z})$ can be described topologically as follows. If C is an irreducible curve then the restriction $H^2(S, \mathbb{Z}) \to H^2(C, \mathbb{Z}) \cong \mathbb{Z}$ gives a linear form on $H^2(S, \mathbb{Z})$ and hence by Poincaré duality an element $c_1(C) \in H^2(S, \mathbb{Z})$. For D a reducible divisor, we define $c_1(D)$ by linearity. Then $c_1(D_1) \cdot c_1(D_2) = D_1 \cdot D_2$ for divisors D_1,D_2 on S. This enables us to define (topologically) an intersection pairing on $\mathrm{Pic}(S)$ by pulling back to $\mathrm{Pic}(S)$ the non-degenerate bilinear intersection form on $H^2(S,\mathbb{Z})$ using the map $c_1 : \mathrm{Pic}(S) \to H^2(S,\mathbb{Z})$. The two definitions of intersection pairing i.e. algebraic and topological, are equivalent [**GH78**, Chap. 4 §1].

Example 1.4. If C is an irreducible divisor on a surface S, then we can define the self-intersection number $C \cdot C$, usually denoted C^2, by $C^2 = \deg_C(\mathcal{O}_S(C) \otimes \mathcal{O}_C)$ [**Har77**, Lem. V.1.3]. This can be extended by linearity to any divisor on S. In particular for $S = \mathbb{P}^2$, then $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}[H] \cong \mathbb{Z}$ where H is the class of a hyperplane. Any two lines are linearly equivalent and any two distinct lines meet at one point, thus we have $H^2 = 1$. This determines the intersection pairing on $\mathbb{P}^2$ by linearity.

Example 1.5 (Canonical line bundle). Let $\Omega_S = \Omega_{S/\mathbb{C}}$ be the sheaf of differential of $S/\mathbb{C}$ and let $\omega_S = \bigwedge^2 \Omega_S$ be the canonical sheaf [**Har77**, II §8]. Any divisor K in the linear equivalence class corresponding to ω_S is called the canonical divisor. The self intersection of the canonical divisor defines a numerical invariant depending only on the surface S. For example, if $S = \mathbb{P}^2$, $K = -3H$ and so $K^2 = 9$. We shall write K_S to denote the canonical divisor class on S and the same notation for the canonical sheaf.

1.2. Nodal Curves on Surfaces

Let d be a non-negative integer and consider the linear system $|dH|$ of curves of degree d on $\mathbb{P}^2$. For each $\delta \geq 0$, consider the subscheme $\widetilde{V}^{d,\delta}$ of $|dH|$ parameterizing the locus of δ-nodal irreducible curves of degree d on $\mathbb{P}^2$. The closure of this scheme $V^{d,\delta}$ in $|dH|$ is called the *Severi variety* of δ-nodal curves in $|dH|$. Severi varieties were introduced by F. Severi [**Sev68**] in his study of irreducibility of moduli space of curves of a given genus. The variety $V^{d,\delta}$ is irreducible - a fact that was first asserted by Severi and proven by J. Harris [**Har86**]. The notion of Severi varieties can be extended to any smooth projective surface S. Let L be an effective line bundle i.e. such that $|L|$ is not empty. The locus of δ-nodal irreducible curves in $|L|$ is locally closed and its closure $V^{(S,L),\delta}$ is called the Severi variety of δ-nodal curves in $|L|$.

There are several questions that one can ask about the Severi varieties e.g. questions about their irreducibility and further the nature of their irreducible components, questions about the dimensions of the components as well as questions about their degrees in the projective space $|L|$. The problem of finding the degree of the Severi variety is enumerative in nature and reduces to the question: *how many elements of* $V^{(S,L),\delta}$ *pass through* $\dim|L|-\delta$ *general points on* S*?* The answer to such a question is called the *Severi degree*. We shall denote the Severi degree by $n^{(S,L),\delta}$.

On $\mathbb{P}^2$ we shall write by $n^{d,\delta} := n^{(\mathbb{P}^2,dH),\delta}$. The following results had already been obtained in the 19th century by using classical methods of enumerative geometry.

$n^{d,1} = 3(d-1)^2, d \geq 3,$	J. Steiner - 1848,
$n^{d,2} = \frac{3}{2}(d-1)(d-2)(3d^2-3d-11), d \geq 4,$	A. Cayley - 1863,
$n^{d,3} = \frac{9}{2}d^6 - 27d^5 + \frac{9}{2}d^4 + \frac{423}{2}d^3 - 229d^2 - \frac{829}{2}d + 525, d \geq 4,$	S. Roberts - 1875.

A list with more values of $n^{d,\delta}$ for $\delta \leq 6$ and is provided in [**DFI95**, Prop. 2]. Already evident from the table above is that $n^{d,\delta} \in \mathbb{Q}[d]$ for small values of d and δ. Di Francesco and Itzykson [**DFI95**] conjectured that whenever d is sufficiently large compared to δ, then $n^{d,\delta}$ is a rational polynomial in d and has degree 2δ.

In this thesis, we shall focus on a slightly more general version of the Severi degree. Fix a pair (S,L) of a smooth projective surface and a line bundle and for every $\delta \geq 0$, let $p_1, \ldots, p_{\dim|L|-\delta}$ be a configuration of general points on S. The guiding question shall be the following. *How many reduced but not necessarily irreducible* δ*-nodal curves in* $|L|$ *pass through all the points* p_i. Assuming that this number is finite, it is also called the Severi degree and also denoted by $n^{(S,L),\delta}$. In the sequel we shall assume that there are finitely many δ-nodal possibly reducible curves in $|L|$ through a sufficiently many general point configuration. In fact by [**KST11**, Prop. 2.1], this is the case whenever L is δ-very ample. It is shown there that if L is δ-very ample, then a general δ-dimensional linear system $\mathbb{P}^\delta \subset |L|$ contains a finite number of δ-nodal curves appearing with multiplicity 1 and all other curves in $\mathbb{P}^\delta$ are reduced with geometric genus $\hat{g} > \frac{L(L+K_S)}{2} + 1 - \delta$.

Vainsencher [**Vai95**], proved that for $\delta \geq 6$ then $n^{(S,L),\delta}$ if given by polynomials in the intersection numbers L^1, LK_S, K_S^2 and $c_2(S)$. Göttsche [**Göt98**], gave more general conjectures about the polynomiality of the Severi degrees.

CONJECTURE 1.6. [**Göt98**, Conj. 2.1] *For every $\delta \geq 0$ there exist a universal polynomial $T_\delta \in \mathbb{Q}[x, y, z, w]$ of degree δ such that for every pair (S, L) of a smooth projective surface S and a line bundle L on S then*

$$n^{(S,L),\delta} = T_\delta(L^2, LK_S, K_S^2, c_2(S)) \tag{1.5}$$

whenever L is sufficiently ample with respect to δ.

Tzeng [**Tze12**] gave a proof of Conjecture 1.6 above using algebraic cobordism theory of pairs of line bundles on surfaces and degenerations. Independently, Kool, Shende and Thomas [**KST11**] gave a proof of the conjecture by using techniques in the study of Hilbert schemes of points of curves on surfaces, BPS calculus and using the computations of certain tautological integrals on Hilbert schemes. We shall discuss briefly the proof by Kool, Shende and Thomas in §1.4.2 below. Using combinatorial tools from tropical geometry, S. Fomin and G. Mikhalkin [**FM10**] established the polynomiality of $n^{(S,L),\delta}$ for a large class of toric surfaces.

1.3. Universal Node Polynomials

The term *universal polynomial* in Conjecture 1.6 above means that the polynomial T_δ is independent of the pair (S, L). For a fixed pair (S, L) and $\delta \geq 0$, we write $n_\delta(S, L) = T_\delta(L^2, LK_S, K_S^2, c_2(S))$ to denote the polynomial and in particular write $n_\delta(d) := n_\delta(\mathbb{P}^2, dH)$. In accordance with the terminology used by Kleiman and Piene [**KP04, KP99**], the polynomials $n_\delta(S, L)$ are referred to as *node polynomials*. Note in particular that for the pair $(\mathbb{P}^2, dH)$ one has $n_\delta(d) = T_\delta(d^2, -3d, 9, 3)$ and consequently the conjecture says that $n^{d,\delta}$ is given by a polynomial of degree 2δ for d large enough.

The problem of counting nodal curves on $\mathbb{P}^2$ was solved by Caporaso and Harris [**CH98**]. They defined the generalized Severi degrees $n^{d,\delta}(\alpha, \beta)$ which counts δ-nodal curves in $\mathbb{P}^2$ that satisfy some tangential conditions with respect to a fixed line E and also pass through a sufficient number of general points on $\mathbb{P}^2$ and used deformation theory to introduce recursive formulas for the computation of $n^{d,\delta}(\alpha, \beta)$. Knowing all values of $n^{d,\delta}$ means that all coefficients of $n_\delta(d)$ can be computed by solving linear equations.

The generating function for the node polynomials is expressed in terms of quasi modular forms. We take a small detour to recall some basic facts about modular forms. For more details, one may consult [**BvdGHZ08**]. Let $\mathcal{H} := \{\tau \in \mathbb{C} : \Im(\tau) > 0\}$ be the upper half complex plane and let G be the full modular group $SL(2,\mathbb{Z})/\{\pm I_2\}$. $\mathcal{H}$ admits an action $G \times \mathcal{H} \to \mathcal{H}$ given by

$$\gamma(z) := (\gamma, z) \mapsto \frac{az+b}{cz+d} \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

For $\tau \in \mathcal{H}$ write $q := e^{2\pi i \tau}$. We denote by D the differential operator $D := \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$.

Definition 1.7. Let k be an integer. A holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is modular form of weight k for G if it satisfies the following transformation property

$$f(\gamma(z)) = (cz+d)^k f(z) \text{ for all } z \in \mathcal{H} \text{ and all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \tag{1.6}$$

and is also holomorphic at $i\infty$. Since G is generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.7}$$

then the transformation property (1.6) above is equivalent to saying that $f(z) = f(z+1)$ i.e. f is periodic and $f(-1/z) = z^k f(z)$. In particular $f(z) = f(z+1)$ implies that f has a convergent Fourier series expansion $f(\tau) = \sum_{n\geq 0} a_n q^n$ at $i\infty$.

Example 1.8 (The Eisenstein series). For $k \geq 0$ write $\sigma_k(n) := \sum_{d|n} d^k = \sum_{d|n} (n/d)^k$ and let B_k denote the k-th Bernoulli number. The Eisenstein series is defined by the following series

$$G_k(q) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \qquad k \geq 2. \tag{1.8}$$

G_k is a modular form of weight k for $k = 4, 6, 8, \ldots$.

Example 1.9. The discriminant Δ is a modular form of weight 12. Δ is given by the series

$$\Delta(q) = q \prod_{n=1}^{\infty} (1-q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \cdots. \tag{1.9}$$

DEFINITION 1.10. An almost holomorphic modular form F of weight k is a function $F : \mathcal{H} \to \mathbb{C}$ satisfying the transformation property (1.6) above but which as a function of τ is a polynomial in $(\Im\tau)^{-1}$ with coefficients that are holomorphic functions of q i.e. F has the form

$$F(\tau) = \sum_{m=0}^{M} f_m(\tau)(\Im\tau)^{-m} \tag{1.10}$$

where $f_m(\tau)$ is holomorphic for all $m = 0, 1, \ldots, M$.

The holomorphic function $f_0(\tau)$ obtained formally as the constant term with respect to $(\Im\tau)^{-1}$ of F is called a quasi-modular form of weight k. An example of a quasi-modular form is the Eisenstein series G_2 which from (1.8) above is given by the series

$$G_2(q) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n)q^n. \tag{1.11}$$

We now switch back to the discussion about the generating functions for the node polynomials. First note that

$$DG_2(q) = q + \sum_{n=2}^{\infty} n\sigma_1(n)q^n \text{ and } D^2G_2(q) = q + \sum_{n=2}^{\infty} n^2\sigma_1(n)q^n. \tag{1.12}$$

Consequently DG_2/q and D^2G_2/q are power series whose constant term is 1 and are therefore invertible in $\mathbb{Q}[\![q]\!]$.

THEOREM 1.11. [**Göt98**, Conj. 2.4], [**Tze12**, Thm. 1.2] *There exists universal power series $B_1, B_2 \in \mathbb{Q}[\![q]\!]$ such that*

$$\sum_{\delta\geq 0} n_\delta(S,L)(DG_2(q))^\delta = \frac{(DG_2(q)/q)^{\chi(L)}B_1(q)^{K_S^2}B_2(q)^{LK_S}}{(\Delta(q)D^2G_2(q)/q^2)^{\chi(\mathcal{O}_S)/2}}. \tag{1.13}$$

$B_i(q)$ in (1.13) above are power series in q whose first term is 1. Let $g(t)$ be the compositional inverse of $DG_2(q)$ i.e. the unique power series in $\mathbb{Q}[\![t]\!]$ such that $DG_2(g(t)) = t$ and $g(DG_2(q)) = q$. Denote $g'(t) := \frac{\partial g}{\partial t}$. In particular we have

$$D(DG(g(t))) = \frac{g(t)}{g'(t)}\frac{\partial DG_2(g(t))}{\partial t} = \frac{g(t)}{g'(t)}. \tag{1.14}$$

Using this in (1.13) above yields

$$\sum_{\delta\geq 0} n_\delta(S,L)t^\delta = (t/g(t))^{\chi(L)}B_1(g(t))^{K_S^2}B_2(g(t))^{LK_S}\left(\frac{g(t)g'(t)}{\Delta(g(t))}\right)^{\chi(\mathcal{O}_S)/2}. \tag{1.15}$$

Using the standard formula $\chi(L) = (L^2 - LK_S)/2 + \chi(\mathcal{O}_S)$ and Noether's formula $\chi(\mathcal{O}_S) = (K_S^2 + c_2(S))/12$ (see [**Bea96**, I.14]) we get in particular the following.

PROPOSITION 1.12. [**Göt98**, Prop 2.3] *There exists power series $A_1, \dots, A_4 \in \mathbb{Q}[\![t]\!]$ such that*

$$\mathcal{N}(S,L) := \sum_{\delta=0}^{\infty} n_\delta(S,L)t^\delta = A_1(t)^{L^2} A_2(t)^{K_S^2} A_3(t)^{LK_S} A_4(t)^{c_2(S)}. \tag{1.16}$$

Equation (1.16) above is called the *multiplicativity* of the generating function $\mathcal{N}(S,L)$ in the intersection numbers of the pair (S,L).

1.4. Refined Curve Counting

1.4.1. The Hirzebruch characteristic classes. The main references for this subsection are the Hirzebruch's books [**Hir95, HBJ92**]. Let X be a smooth complex projective variety and $\mathcal{F}$ a coherent sheaf over X. We will see that the Hirzebruch χ_y-genus establishes a connection between the Euler characteristic, the arithmetic genus and the signature of X as a smooth manifold. Recall that the Euler-Poincaré characteristic (In short Euler characteristic) of $\mathcal{F}$ over X is the alternating sum

$$\chi(X,\mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{C}} H^i(X,\mathcal{F}). \tag{1.17}$$

By Serre's finiteness theorem ([**Har77**, Thm. III.5.2]) $H^i(X,\mathcal{F})$ is a finite dimensional vector space over $\mathbb{C}$. Additionally, $H^i(X,\mathcal{F}) = 0$ for all $i > \dim X$ and as a consequence $\chi(X,\mathcal{F})$ is a finite integer. When there's no confusion we shall write $\chi(\mathcal{F})$ to denote the Euler characteristic of $\mathcal{F}$. $\chi(\mathcal{F})$ can be expressed in terms of Chern classes of $\mathcal{F}$ and T_X the tangent sheaf of X. The Todd class and the Chern character of $\mathcal{F}$ are defined respectively by

$$td(\mathcal{F}) = \prod_{i=1}^{r} \frac{\alpha_i}{1 - e^{-\alpha_i}}, \text{ and } ch(\mathcal{F}) = \sum_{i=1}^{r} e^{\alpha_i} \tag{1.18}$$

where α_i are the Chern roots of $\mathcal{F}$ and r is the rank of $\mathcal{F}$ ([**Har77**, Appendix A, §4]).

REMARK 1.13. The arithmetic genus of projective variety X is defined to be

$$p_a(X) = (-1)^n \left(\chi(\mathcal{O}_X) - 1\right). \tag{1.19}$$

Note that since X is a variety then $H^0(X, \mathcal{O}_X) = \mathbb{C}$, therefore by (1.17) and (1.19) above we have

$$p_a(X) = \sum_{i=1}^{n} (-1)^{n+i} \dim_{\mathbb{C}} H^i(X, \mathcal{O}_X) = \sum_{i=0}^{n-1} (-1)^i \dim_{\mathbb{C}} H^{n-i}(X, \mathcal{O}_X) \tag{1.20}$$

where $n = \dim X$. In particular if X is a curve then $p_a(X) = \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X)$.

THEOREM 1.14. [**Har77**, Appendix A, Thm 4.1] *Let $\mathcal{F}$ be a locally free sheaf of rank r on a smooth projective variety X of dimension n then*

$$\chi(\mathcal{F}) = \int_X td(T_X) \cdot ch(\mathcal{F}) \cap [X]. \tag{1.21}$$

This is the Hirzebruch-Riemann-Roch established by Hirzebruch [**Hir95**] for complex varieties and generalized to arbitrary nonsingular varieties over algebraically closed fields by Grothendieck [**Ser56**].

DEFINITION 1.15. The Grothendieck group of an abelian category $\mathcal{A}$ is the free abelian group generated by isomorphism classes of objects in $\mathcal{A}$ modulo the relation $[B] = [A] + [C]$ for every short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$. Denote by $K^0(X)$ the Grothendieck group of the abelian category whose objects are the locally free sheaves on a scheme X. In this context, $K^0(\cdot)$ is a functor that associates to a scheme X to the abelian group $K^0(X)$.

DEFINITION 1.16. [**GS14**, §3.1] A normalized multiplicative genus (see also [**Hir95, HBJ92**]) is a natural transformation of functors $\Phi : K^0(\cdot) \to H^*(\cdot, \Lambda)$ (where Λ is a commutative ring) such that

(a) $\Phi(\mathbb{C}) = 1$ for the trivial bundle,
(b) for a sum $\mathcal{E} \oplus \mathcal{F}$ of two vector bundles then $\Phi(\mathcal{E} \oplus \mathcal{F}) = \Phi(\mathcal{E})\Phi(\mathcal{F})$ and
(c) to every such genus Φ corresponds a power series $f_\Phi \in 1 + z\Lambda[\![z]\!]$ called the *characteristic power series,* such that for a line bundle L then $\Phi(L) = f_\Phi(c_1(L))$.

EXAMPLE 1.17. We are interested in particular on the Hirzebruch χ_y-genus. It is given for a smooth complex projective variety X and a holomorphic vector bundle

$\mathcal{E}$ over X by the alternating sum

$$\chi_y(X,\mathcal{E}) = \sum_{p,q\geq 0}\Big((-1)^q \dim_{\mathbb{C}} H^q(X,\mathcal{E}\otimes\Omega_X^p)\Big)y^p = \sum_{p\geq 0}\chi(X,\mathcal{E}\otimes\Omega_X^p)y^p \tag{1.22}$$

where $\Omega_X^p := \bigwedge^p \Omega_X$ is the p-th exterior power of the holomorphic cotangent bundle on X. According to the *generalized Hirzebruch-Riemann-Roch* [**Hir95**, §21.3] theorem

$$\chi_y(X,\mathcal{E}) = \int_X T_y^*(T_X)\cdot ch_{(1+y)}(\mathcal{E})\cap [X] \tag{1.23}$$

here

$$ch_{(1+y)}(\mathcal{E}) = \sum_{i=1}^{r} e^{\beta_i(1+y)} \text{ and } T_y^*(T_X) = \prod_{i=1}^{n}\frac{\alpha_i(1+ye^{-\alpha_i(1+y)})}{1-e^{-\alpha_i(1+y)}} \tag{1.24}$$

where r is the rank of $\mathcal{E}$, n is the dimension of X, β_i are the Chern roots of $\mathcal{E}$ and α_i are the Chern roots of the tangent bundle T_X. The power series $T_y^*(T_X)$ is called the modified Todd class.

Lemma 1.18. [**SY07**, §6] *The Hirzebruch χ_y-genus unifies the Euler Poincaré characteristic, the signature and the Euler number.*

Proof. It is already evident from (1.22) above that $\chi(X,\mathcal{E}) = \chi_0(X,\mathcal{E})$. Writing

$$\chi_y(X,\mathcal{E}) = \chi(X,\mathcal{E}) + \sum_{p\geq 1}\chi(X,\mathcal{E}\otimes\Omega_X^p)y^p$$

and substituting $y=0$ proves the assertion. Alternatively substituting $y=0$ in (1.24) above and using it in (1.23) then the assertion follows by the Hirzebruch-Riemann-Roch theorem. Substituting $y=1$ in the modified Todd class (1.24) above one obtains the total Thom-Hirzebruch L-class

$$L_1^*(T_X) = \prod_{i=1}^{n}\left(\frac{\alpha_i}{\tanh \alpha_i}\right) \text{ therefore } \chi_{(1)}(X,\mathcal{O}_X) = \int_X L_1^*(T_X)\cap [X] \tag{1.25}$$

which is the signature of X(Hirzebruch-signature theorem). Writing

$$Q(y,x) = \frac{x(y+1)}{1-e^{-x(y+1)}} - yx = \frac{x(1+ye^{-x(1+y)})}{1-e^{-x(1+y)}}. \tag{1.26}$$

Then by [**Hir95**, Lem. 1.8.1] we have $Q(-1,x) = 1+x$. In this case, the modified Todd class specializes to the total Chern class therefore, the χ_y-genus determines the euler number at $y=-1$. $\square$

Remark 1.19. In the later parts of this thesis we shall use a slightly modified version of the χ_y-genus. Let $X = \mathbb{P}^N$ then

$$h^{p,q}(\mathbb{P}^N) = \begin{cases} 1 \text{ if } 0 \leq p = q \leq N \\ 0 \text{ if } p \neq q. \end{cases} \tag{1.27}$$

Replacing y by $-y$ then one obtains

$$\chi_{-y}(\mathbb{P}^N) = \sum_{p=0}^{N} y^p = \frac{y^{N+1}-1}{y-1}. \tag{1.28}$$

This example motivates the definition following *quantum number* $[d]_k$ which will play an important role in the later parts of this thesis. Define $\widetilde{\chi_{-y}}(\cdot) = y^{-\dim(\cdot)/2}\chi_{-y}(\cdot)$, then it is easy to see that

$$[d]_k := \widetilde{\chi_{-y}}(\mathbb{P}^{d-1}) = \frac{y^{d/2} - y^{-d/2}}{y^{1/2} - y^{-1/2}}. \tag{1.29}$$

In [**GS14**] the χ_y-genus defined above is used to introduce the *refined invariants* for a pair (S, L) of a smooth projective surface and a line bundle L on S. The refined invariants are conjecturally expressed by universal polynomials called the *refined node polynomials*. Furthermore a conjecture is stated asserting that the generating function for the refined node polynomials satisfy a multiplicative structure analogous to the one in (1.16) above. We need the following elementary facts about the relative Hilbert schemes before introducing the refined invariants.

Definition 1.20. Let B be a scheme. A family of projective varieties over B is a flat projective morphism of schemes $f : F \to B$. We shall denote this by F/B.

The condition of flatness of a morphism is useful in the following sense: if $f : X \to Y$ is a flat morphism of schemes and Y is connected then many properties of the fibers $X_y := f^{-1}(y)$ are independent of the choice of $y \in Y$. In particular a family of closed subschemes of a projective space over a reduced connected base B is flat if and only if all fibers have the same Hilbert polynomial [**EH00**, Prop. III-56]. The Hilbert polynomial of a subscheme $Z \subset \mathbb{P}^r$ is a polynomial whose degree is equal to the dimension of the subscheme Z [**EH00**, Prop. III-59].

Let X be a smooth quasi-projective variety over $\mathbb{C}$ and X/B a flat family over B. Consider the contravariant functor $\mathcal{H}ilb^n_{X/B} : (Sch) \to (Sets)$ that associates to every scheme T to the set

$$\mathcal{H}ilb^n_{X/B}(T) = \{Z \subset X \times_B T \text{ flat over } T \text{ with Hilbert polynomial } n\}. \tag{1.30}$$

We can restate (1.30) above with the help of the following commutative diagram

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "\pi"'] & X \times_B T \arrow[d, "p_2"] \\ & T \end{tikzcd}$$

where $\pi : Z \to T$ is flat with every fiber Z_t having a constant Hilbert polynomial n. Thus the T-valued points of $\mathcal{H}ilb^n_{X/B}$ are therefore closed subschemes $Z \subset X \times_B T$ that are finite over T and such that $\pi_* \mathcal{O}_Z$ is a locally free module of $\mathcal{O}_T$ of rank n. By a theorem of Grothendieck (explained in [**FGI$^+$05**]), $\mathcal{H}ilb^n_{X/B}$ is representable by a projective scheme $Hilb^n_{X/B}$.

Remark 1.21. If $B = Spec(k) = pt$ for k a field then $Hilb^n_{X/k}$ is the collection of zero dimensional subschemes $Z \subset X$. We denote $X^{[n]} := Hilb^n_{X/k}$ and call it the Hilbert scheme of points on X.

Let (S, L) be a pair of smooth projective surface over $\mathbb{C}$ and L a line bundle on S. Let $\mathbb{P}^\delta$ be a general δ-dimensional subspace of the complete linear system $|L|$ and consider the universal curve

$$\mathcal{C} = \{(p, [C]) : p \in C\} \subset S \times \mathbb{P}^\delta. \tag{1.31}$$

The natural projection to $\mathbb{P}^\delta$ defines a family $\mathcal{C} \to \mathbb{P}^\delta$. For $i \geq 1$, let $Hilb^i_{\mathcal{C}/\mathbb{P}^\delta}$ be the corresponding relative Hilbert scheme. If L is δ-very ample then [**KST11**, Thm. 3.4] says that $N^{(S,L),\delta}$ is a linear combination of $\chi(Hilb^i_{\mathcal{C}/\mathbb{P}^\delta})$ for $i = 1, \ldots, \delta$. Precisely, it is the coefficient $n_{g-\delta}$ in the generating series

$$q^{1-g} \sum_{i=0}^{\infty} \chi(Hilb^i_{\mathcal{C}/\mathbb{P}^\delta}) q^i = \sum_{r=g-\delta}^{g} n_r q^{1-r} (1-q)^{2r-2}. \tag{1.32}$$

By a theorem of Ellingsrud, Lehn and Göttsche [**EGL01**] then one can show that the Euler characteristic $\chi\big(Hilb^n_{\mathcal{C}/\mathbb{P}^\delta}\big)$ can be expressed in a universal way in terms of L^2, K_S^2, LK_S and $c_2(S)$. This is used in [**KST11**, Thm. 4.1] to prove that whenever L is δ-very ample then

$$N^{(S,L),\delta} = T_\delta(L^2, LK_S, K_S^2, c_2(S)) \tag{1.33}$$

for some universal polynomial $T_\delta \in \mathbb{Q}[x,y,z,w]$. This shows that indeed, the number of δ-nodal curves on a surface is a topological invariant of the pair (S,L).

1.4.2. Refined Invariants. Let (S,L) be a pair of smooth projective surface over $\mathbb{C}$ and L a line bundle on S. By a theorem of Forgarty [**Fog68**], the Hilbert scheme of n points $S^{[n]}$ on S is smooth projective irreducible variety of dimension $2n$. Denote by $T_{S^{[n]}}$ the tangent bundle of $S^{[n]}$ and let $t_1,\dots,t_{2n}$ be the Chern roots of $T_{S^{[n]}}$. Let $\mathcal{Z}_n(S) \subset S\times S^{[n]}$ be the universal family. Then there exists projections

$$\begin{array}{ccc} \mathcal{Z}_n(S) & \xrightarrow{\sigma} & S^{[n]} \\ {\scriptstyle q}\downarrow & & \\ S. & & \end{array}$$

Note that the fiber of σ over a point $[Z]\in S^{[n]}$ is isomorphic to the subscheme Z i.e.

$$\sigma^{-1}([Z]) = \{(p,[Z]) \in S\times S^{[n]} : p\in Z\} \cong Z.$$

Thus σ is a flat morphism and hence the sheaf $L^{[n]} := \sigma_* q^* L$ on $S^{[n]}$ is locally free of rank n. Denote by $l_1,\dots,l_n$ the Chern roots of $L^{[n]}$.

Proposition 1.22. [**GS14**, Prop. 47] *Suppose that $Hilb^n(\mathcal{C}/\mathbb{P}^\delta)$ is nonsingular for all n. Then*

(1.34)

$$\chi_{-y}(Hilb^n(\mathcal{C}/\mathbb{P}^\delta)) = \operatorname*{res}_{x=0}\left[\left(\frac{Q(-y,x)}{x}\right)^{\delta+1}\int_{S^{[n]}}\prod_{i=1}^{2n}Q(-y,t_i)\prod_{j=1}^{n}\left(\frac{l_j}{Q(-y,l_j+x)}\right)\right]$$

where $Q(\cdot,\cdot)$ is the power series in (1.26) above.

Remark 1.23. By definition,

$$\prod_{i=1}^{2n}Q(-y,t_i)\prod_{j=1}^{n}\left(\frac{l_j}{Q(-y,l_j+x)}\right) \in H^*(S^{[n]},\mathbb{Q})[y][\![x]\!].$$

Therefore, the term in square brackets on the right hand side of (1.34) is a Laurent series in x with coefficients in $\mathbb{Q}[y]$. The generating function for the $\chi_{-y}(Hilb^n(\mathcal{C}/\mathbb{P}^\delta))$ satisfies the following.

PROPOSITION 1.24. [**BG16**, Prop. 2.2] *Assume that $Hilb^n_{\mathcal{C}/\mathbb{P}^\delta}$ is nonsingular for all n. Then there exists polynomials $n_0(y), \dots, n_g(y)$ such that*

$$\sum_{n=0}^{\infty} \chi_{-y}(Hilb^n_{\mathcal{C}/\mathbb{P}^\delta})t^n = \sum_{r=0}^{g} n_r(y)t^r((1-t)(1-ty))^{g-r-1} \tag{1.35}$$

where $g := g(L)$ is the arithmetic genus of a curve in $|L|$.

A necessary condition for $Hilb^n_{\mathcal{C}/\mathbb{P}^\delta}$ to be smooth for all n is that L is δ-very ample [**GS14**, Theorem 41]. The $\chi_{-y}(Hilb^n_{\mathcal{C}/\mathbb{P}^\delta})$ is a topological invariant of the pair (S, L), therefore, the coefficients $n_r(y)$ on the right hand side of (1.35) are also topological invariants of (S, L) and should have a topological interpretation.

DEFINITION 1.25. [**BG16**, Defn. 2.4] Suppose that L is δ-very ample line bundle on S. Then the *refined invariants* of S and L are defined to be

$$\widetilde{N}^{(S,L),\delta}(y) := n_\delta(y)/y^\delta \tag{1.36}$$

where $n_\delta(y)$ is the polynomial in (1.35) above.

Conjecturally, the refined invariants defined above are given by universal polynomials in the intersection numbers of (S, L). The following is an analogous reformulation of Theorem 1.6 above.

THEOREM 1.26. *For every $\delta \geq 0$ there exists a universal polynomial $T_\delta \in \mathbb{Q}[y^{\pm 1}][q, r, s, t]$ such that for every pair (S, L) then*

$$\widetilde{N}^{(S,L),\delta}(y) = T_\delta(L^2, LK_S, K_S^2, c_2(S)) \in \mathbb{Q}[y^{\pm 1}] \tag{1.37}$$

whenever L is sufficiently ample with respect to δ.

For a fixed pair (S, L) of a smooth projective surface and a line bundle and every $\delta \geq 0$ denote by $\widetilde{N}_\delta(S, L; y)$ the corresponding universal polynomial and call it the *refined polynomial.* We consider again the generating function for the refined polynomials. Let

$$\widetilde{\mathcal{N}}(S, L; y) = \sum_{\delta \geq 0} \widetilde{N}_\delta(S, L; y)t^\delta \tag{1.38}$$

be the generating function. An analogous statement to Theorem 1.11 is stated in [**GS14**] i.e. $\mathcal{N}(y; S, L)$ has a multiplicative structure. To state the analogous conjecture we require *refined versions* of the quasi-modular forms (1.9) and (1.12) above. Let $D := q\frac{\partial}{\partial q}$ be a differential operator and let

$$\begin{aligned} \widetilde{\Delta}(y,q) &:= q\prod_{n=1}^{\infty}(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2, \\ \widetilde{DG_2}(y,q) &:= \sum_{n=1}^{\infty} q^n \sum_{d|n} [d]_y^2 \frac{n}{d}. \end{aligned} \tag{1.39}$$

CONJECTURE 1.27. [**GS14**, Conj. 62] *There exists universal power series* $B_1, B_2 \in \mathbb{Q}[y, y^{-1}][\![q]\!]$ *such that*

$$\sum_{\delta \geq 0} \widetilde{N}_\delta(S, L; y)\widetilde{DG_2}(y,q)^\delta = \frac{(\widetilde{DG_2}(y,q)/q)^{\chi(L)} B_1(y,q)^{K_S^2} B_2(y,q)^{LK_S}}{(\widetilde{\Delta}(y,q) D\widetilde{DG_2}(y,q)/q^2)^{\chi(\mathcal{O}_S)/2}}. \tag{1.40}$$

Again every power series appearing in (1.40) above equation is invertible. Using similar arguments as in §1.3 above we obtain the following refined analogue of Theorem 1.16 above.

CONJECTURE 1.28. [**GS14**] *There exists universal power series* $A_i(y,t) \in \mathbb{Q}[y^{\pm 1}][\![t]\!], i = 1,2,3,4$ *such that for all pairs* (S, L) *of a smooth projective surface and a line bundle we have*

$$\widetilde{\mathcal{N}}(S, L; y) = A_1(y,t)^{L^2} A_2(y,t)^{LK_S} A_3(y,t)^{K_S^2} A_4(y,t)^{c_2(S)}. \tag{1.41}$$

Again as in §1.3, (1.41) above is called the multiplicativity of the generating function for the refined node polynomials. One of the main results in this thesis is a result on the multiplicativity structure of the generating function for the refined node polynomials on a particular subclass of toric surfaces. In the next section, we introduce the refined Severi degrees via a modification of the Caporaso - Harris recursion [**CH98**]. For the considered cases of toric surfaces, their refined Severi degrees are conjecturally equal to the refined invariants and thus they are also given by the universal polynomials.

1.4.3. Refined Caporaso-Harris Recursion. Refined Severi degrees were introduced in [**GS14**, §5.1] via a formal refinement of the Caporaso-Harris recursion formula [**CH98**]. Their relation to the refined invariants discussed in the previous subsection is conjectural(Conjecture 1.31).

Let $\alpha = (\alpha_1, \alpha_2, \dots)$ and $\beta = (\beta_1, \beta_2, \dots)$ be a pair sequences of non-negative integers with a finite support. For a sequence α, define $\|\alpha\| = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots$ and $|\alpha| = \alpha_1 + \alpha_2 + \cdots$. In [**CH98**], the *relative Severi degree* $n^{d,\delta}(\alpha,\beta)$ is defined to be the number of δ-nodal curves of degree d on $\mathbb{P}^2$ passing through a configuration of $d(d+3)/2 - \delta - |\alpha|$ generic points and in addition, have tangency of order i at α_i fixed points of a fixed line L for each i and tangency of order i at some β_i points on L for each i. Vakil [**Vak00**], generalized the methods of Caporaso and Harris to extend the definition of the relative Severi degrees to rational ruled surfaces.

Definition 1.29 below is a modified version of Vakil's formulation [**Vak00**, Thm. 1.3]. Let S be $\mathbb{P}^2$ or a rational ruled surface Σ_m and L a line bundle on S. On $\mathbb{P}^2$ let E be a line and H the hyperplane bundle. On Σ_m let E be the class of section with $E^2 = -m$ and define $H := E + mF$ where F is the class of the fiber of the ruling. Let α, β be sequences such that $\|\alpha\| + \|\beta\| = EL$. For every $\delta \geq 0$ define

$$\gamma(L, \beta, \delta) := \dim |L| - EL + |\beta| - \delta.$$

Definition 1.29. [**GS14**, Recursion 71, Prop. 73] The *refined relative Severi* degrees denoted by $N^{(S,L),\delta}(\alpha, \beta; y)$ are defined by the following recursive formula. If $\gamma(L, \beta, \delta) > 0$ then

$$\begin{aligned} N^{(S,L),\delta}(\alpha, \beta; y) = &\sum_{k:\beta_k>0} [k]_y N^{(S,L),\delta}(\alpha + e_k, \beta - e_k; y) + \\ &\sum_{\alpha',\beta',\delta'} \prod_i [i]_y^{\beta_i'-\beta_i} \binom{\alpha}{\alpha'} \binom{\beta'}{\beta} N^{(S,L-H),\delta'}(\alpha', \beta'; y), \end{aligned} \tag{1.42}$$

where $e_k = (0, \dots, 0, 1, 0, \dots)$ i.e. the sequence whose entries are all zero except at position k where the entry is 1. The second sum runs through all α', β', δ' satisfying

$$\begin{aligned} &\alpha' \leq \alpha, \beta' \geq \beta, \\ &\|\alpha'\| + \|\beta'\| = E(L - E), \\ &\delta' = \delta - E(L - E) + |\beta' - \beta|, \end{aligned} \tag{1.43}$$

and subject to the following initial conditions:

(1) if $\gamma(L, \beta, \delta) < 0$ then $N^{(S,L),\delta}(\alpha, \beta; y) = 0$,
(2) if $\gamma(L, \beta, \delta) = 0$ then $N^{(S,L),\delta}(\alpha, \beta; y) = 0$ unless
   (a) $S = \mathbb{P}^2$ then put $N^{(\mathbb{P}^2,H),0}(1, 0; y) = 1$,
   (b) $S = \Sigma_m$ then put $N^{(\Sigma_m,kF),0}(k, 0; y) = 1$ for $k \geq 0$.

Denote by $N^{(S,L),\delta}(y) := N^{(S,L),\delta}(0, LE; y)$. We call $N^{(S,L),\delta}(y)$ the *refined Severi degree* (non-relative).

The recursive formula (1.42) above has been generalized in [**BG16**, §7] to include the case $S = \mathbb{P}(1,1,m)$. In fact the recursion of $L = dH$ on $\mathbb{P}(1,1,m)$ is identical to that of $L = dH$ on Σ_m. The formula (1.42) has been chosen so that it specializes at $y = 1$ to the usual recursive formula of Caporaso and Harris [**CH98**] for $S = \mathbb{P}^2$ and to the more general recursive formula of Vakil [**Vak00**] for $S = \Sigma_m$. Thus in particular we have

$$N^{(S,L),\delta}(\alpha, \beta; 1) = n^{(S,L),\delta}(\alpha, \beta) \text{ and } N^{(S,L),\delta}(1) = n^{(S,L),\delta}.$$

Remark 1.30. We have written a Maple program that incorporates this recursion. In Chapter 3 of this thesis, extensive computations with this program are used. We shall use tropical methods (in Chapter 2) to give an alternative definition (non-recursive) of the refined Severi degrees.

The following conjecture relates the refined Severi degrees defined by to the refined invariants $\widetilde{N}^{(S,L),\delta}(y)$ as given in Definition 1.25 above.

Conjecture 1.31. [**GS14**, Conj. 75] *Let $S = \mathbb{P}^2$ or a rationally ruled surface and let L be a line bundle on S. Let $\mathbb{P}^\delta \subset |L|$ be a general δ-dimensional subspace and assume that $\mathbb{P}^\delta$ contains no non-reduced curves and no curves with negative self intersection. Then $N^{(S,L),\delta}(y) = \widetilde{N}^{(S,L),\delta}(y)$. Explicitly:*

(1) on $\mathbb{P}^2$ we have $N^{d,\delta}(y) = \widetilde{N}^{d,\delta}(y)$ for $d \geq \delta/2 + 1$,
(2) assume $c + d > 0$. Then $N^{(\mathbb{P}^1\times\mathbb{P}^1, cF+dH),\delta}(y) = \widetilde{N}^{(\mathbb{P}^1\times\mathbb{P}^1, cF+dH),\delta}(y)$ for $c, d \geq \delta/2$,
(3) on $S = \Sigma_m$ with $m > 0$, assume $c + d > 0$. Then $N^{(\Sigma_m, cF+dH),\delta}(y) = \widetilde{N}^{(\Sigma_m, cF+dH),\delta}(y)$ for $\delta \leq \min(2d, c)$.

If Conjecture 1.31 above is true then it follows therefore that whenever S is smooth and L is sufficiently ample then the refined Severi degrees $N^{(S,L),\delta}(y)$ are given the refined node polynomials $N_\delta(S, L; y)$, which are polynomials in the intersection numbers $L^2, LK_S, K_S^2, c_2(S)$.

REMARK 1.32. The weighted projective space $\mathbb{P}(1,1,m)$ is singular for $m \geq 2$ and thus Conjecture 1.26 and Conjecture 1.31 do not apply. The refined invariant $\widetilde{N}^{(S,L),\delta}(y)$ has not been defined in this case. The refined Severi degrees $N^{(\mathbb{P}(1,1,m),dH),\delta}(y)$ are compared with the corresponding refined invariants $\widetilde{N}^{(\Sigma_m,dH),\delta}(y)$ on the minimal resolution of Σ_m of $\mathbb{P}(1,1,m)$.

CONJECTURE 1.33. [**BG16**, Conj. 2.15] *There exists a polynomial $N_\delta(d,m;y)$ of degree 2δ in d and of degree δ in m such that $N^{(\mathbb{P}(1,1,m),dH),\delta}(y) = N_\delta(d,m;y)$ for $\delta \leq (2d-2, 2m-1)$.*

Again we consider the generating function for the conjectural *refined node polynomials* of $\mathbb{P}(1,1,m)$ given in Conjecture 1.33 above.

CONJECTURE 1.34. [**BG16**, Conj. 2.16] *There exists power series $C_1, C_2, C_3 \in \mathbb{Q}[y^{\pm 1}][\![q]\!]$ such that*

$$\sum_{\delta=0}^{\infty} N_\delta(d,m;y)\left(\widetilde{DG_2}\right)^{\delta} = \left(\sum_{\delta=0}^{\infty} \widetilde{N}^{(\Sigma_m,dH),\delta}(y)\left(\widetilde{DG_2}\right)^{\delta}\right) C_1^{(m+2)d} C_2^{m+2} C_3. \tag{1.44}$$

1.4.4. Proving the multiplicativity. We have introduced (conjecturally) the refined node polynomials for the pair (S,L) where S is either $\mathbb{P}^2$ or the rational ruled surfaces. Furthermore, we have stated the conjecture that their generating functions are multiplicative in the intersections numbers of (S,L). Consider the formal logarithm of the generating function for the refined node polynomials given by (1.38) above

$$\mathcal{Q}(S,L;y) = \log \mathcal{N}(S,L;y) = \sum_{\delta \geq 1} Q_\delta(S,L;y) t^\delta. \tag{1.45}$$

Assuming Conjecture 1.28 then (1.45) is equivalent to saying that there exists universal power series $a_1(y,t), \ldots, a_4(y,t) \in \mathbb{Q}[y^{\pm 1}][\![t]\!]$ such that

$$Q_\delta(S,L;y) = a_1(y,t)L^2 + a_2(y,t)LK_S + a_3(y,t)K_S^2 + a_4(y,t)\chi(c_2(S)). \tag{1.46}$$

In other words proving the multiplicativity of the generating function is equivalent to proving the $\mathbb{Q}[y^{\pm 1}]$-linearity of $Q_\delta(S,L;y)$ in the intersection numbers. Note also that for $\mathbb{P}(1,1,m)$, (1.44) exhibit a *pseudo-multiplicativity* of the generating function of the conjectural node polynomials. In Chapter 3 we shall use combinatorial methods to exhibit a $\mathbb{Q}[y^{\pm 1}]$-linearity structure of $Q_\delta(S, L; y)$ for $S = \mathbb{P}^2, \Sigma_m$ and $\mathbb{P}(1, 1, m)$.

1.5. Welschinger Numbers

For a detailed treatment of real algebraic varieties one may consult [**DK00, Sil89**]. We include only the bare minimum material necessary for our purposes. The aim here is to state a conjecture asserting that the generating functions for what is called the *Welschinger numbers* satisfy the multiplicative structure similar to (1.41).

Definition 1.35. Let X be a complex analytic variety. A *real structure* c_X on X is an anti-holomorphic involution $c_X : X \to X$ that is differentiable at smooth points of X. A real variety is a pair (X, c_X) where X is a complex variety and c_X is a real structure on X. If c_X is understood in the context we drop it in the notation. The fixed point set of c_X is called the *real part* of X and denoted by $\mathbb{R}X$. A closed subvariety $Z \subset X$ is said to be real if Z is invariant under c_X.

Definition 1.36. Let L be a line bundle on a complex surface S endowed with a complex structure c_X. Assume that $L = \mathcal{O}_X(D)$ for a divisor $D \in \mathrm{Pic}(X)$ and denote by $|L|$ the corresponding general linear system. The real part of $|L|$ denoted by $\mathbb{R}|L|$ consist of real divisors on X linearly equivalent to D.

Consider the problem of counting real curves satisfying a given real point configuration on $\mathbb{RP}^2$. A real point $p \in \mathbb{RP}^2$, as per the definition above, is either a single point invariant under conjugation or is a pair of complex conjugated points. Thus the answer to the counting problem will in general depend on the given configuration. Jean-Yves Welschinger [**Wel03, Wel05**] showed that on a real symplectic four manifold X, then counting real rational J-holomorphic curves (with an appropriate sign) in a given homology class on X, yields an invariant independent of the real point configuration.

Related to the Welschinger invariants are the so called the *Welschinger numbers*. The Welschinger numbers $W^{d,\delta}(\omega)$ count with suitable signs the δ-nodal curves of degree d in $\mathbb{P}^2$ through a configuration ω of $d(d+3)/2 - \delta$ real points and $W^{(S,L),\delta}(\omega)$ counts (with suitable sign) the δ-nodal curves in the linear system $|L|$

on a real algebraic surface S through a configuration ω of $\dim|L|-\delta$ real points. The Welschinger numbers depend in general on the point configuration, however in [**Mik05**], it is shown that for the so called *subtropical* configuration of points, they coincide with the *tropical Welschinger invariants* $W^{d,\delta}_{\mathrm{trop}}, W^{(S,L),\delta}_{\mathrm{trop}}$, defined via tropical geometry. The tropical Welschinger invariants are independent of the tropically generic configuration of points.

We shall assume that a subtropical point configuration has been chosen and write $W^{d,\delta}$ (respectively $W^{(S,L),\delta}$) for $W^{d,\delta}(\omega)$ (respectively $W^{(S,L),\delta}(\omega)$). If S is $\mathbb{P}^2$ or a rational ruled surface, there is a Caporaso-Harris type recursion formula for the Welschinger numbers [**IKS15, IKS09**]. A modified version of the recursion is given in [**GS14**].

Definition 1.37. [**GS14**, Recursion 87] With the same notation, assumptions and initial conditions as in Definition 1.29, the *relative tropical Welschinger invariants* $W^{(S,L),\delta}_{\mathrm{trop}}(\alpha,\beta)$ are given by the following recursion formula: if $\gamma(L,\beta,\delta)>0$,

$$\begin{aligned} W^{(S,L),\delta}_{\mathrm{trop}}(\alpha,\beta) = &\sum_{k\text{ odd}:\beta_k>0} (-1)^{(k-1)/2} W^{(S,L),\delta}_{\mathrm{trop}}(\alpha+e_k,\beta-e_k) \\ &+ \sum_{\alpha',\beta',\delta'} \prod_{i\text{ odd}} ((-1)^{(i-1)/2})^{\beta'_i-\beta_i} \binom{\alpha}{\alpha'}\binom{\beta'}{\beta} W^{(S,L),\delta'}_{\mathrm{trop}}(\alpha',\beta') \end{aligned} \tag{1.47}$$

where the second sum runs through all odd sequences α',β' and all δ' satisfying (1.43). We put $W^{(S,L),\delta}_{\mathrm{trop}} = W^{(S,L),\delta}_{\mathrm{trop}}(0,LE)$ and in particular for $\mathbb{P}^2$ we write $W^{d,\delta}_{\mathrm{trop}} = W^{(\mathbb{P}^2,dH),\delta}_{\mathrm{trop}}$.

From §1.4 above, the refined Severi degrees $N^{(S,L),\delta}(y)$ for a pair (S,L) of a toric surface and a toric line bundle is a Laurent polynomial in y such that

$$N^{(S,L),\delta}(1) = n^{(S,L),\delta}.$$

By Mikhalkin correspondence theorem (this shall be discussed in Chapter 2), the refined Severi degree specializes at $y=-1$ to the Welschinger number $W^{(S,L),\delta}$ (see also [**GS14**, Prop. 88]). This means that the refined Severi degrees unify the Severi degrees and the Welschinger numbers. The formulas (1.39) specialize at $y = -1$ to

$$
\begin{aligned}
\widetilde{\Delta}(-1,q) &:= q\prod_{n=1}^{\infty}(1-q^n)^{20}(1+q^n)^4, \\
\widetilde{DG_2}(-1,q) &:= \sum_{n=1}^{\infty} q^n \sum_{d|n} [d]_{-1}^2 \frac{n}{d}.
\end{aligned} \tag{1.48}
$$

A trivial computation shows that

$$
\begin{aligned}
\widetilde{\Delta}(-1,q) &= \eta(q)^{16}\eta(q^2)^4, \\
\widetilde{DG_2}(-1,q) &= G_2(q) - G_2(q^2) = \sum_{n>0}\left(\sum_{d|n,\ d \text{ odd}} \frac{n}{d}\right) q^n
\end{aligned} \tag{1.49}
$$

where $G_2(q)$ is the Eisenstein series (1.11) and $\eta(q)$ is the Dirichlet eta function given by the power series

$$
\eta(q) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n).
$$

Write $\overline{G}_2(q) := \widetilde{DG_2}(-1,q)$, $D\overline{G}_2(q) = D\widetilde{DG_2}(-1,q)$, and $\overline{B}_i(q) := B_i(-1,q)$ where $B_i(y,q)$ are the power series in (1.40) above. Conjecture 1.27 above specializes as follows.

CONJECTURE 1.38. *There exists universal power series $\overline{B}_1(q), \overline{B}_2(q) \in \mathbb{Q}[\![q]\!]$ such that*

$$
\sum_{\delta=0}^{\infty} W^{(S,L),\delta}\left(\overline{G}_2(q)\right)^{\delta} = \frac{(\overline{G}_2(q)/q)^{\chi(L)}\overline{B}_1(q)^{K_S^2}\overline{B}_2(q)^{LK_S}}{(\eta(q)^{16}\eta(q^2)^4 D\overline{G}_2(q)/q^2)^{\chi(\mathcal{O}_S)/2}}. \tag{1.50}
$$

whenever L is a δ-very ample line bundle on S.

We shall show in Chapter 3 that in particular cases of (S,L) and for δ small enough then (1.50) above implies that

$$
W^{(S,L),\delta} = \operatorname*{Coeff}_{q^{\chi(L)-1}}\left[\frac{\overline{G}_2(q)^{\chi(L)-1-\delta}\overline{B}_1(q)^{K_S^2}}{\overline{B}_2(q)^{-LK_S}}\left(\frac{D\overline{G}_2(q)}{\eta(q)^{16}\eta(q^2)^4}\right)^{\chi(\mathcal{O}_S)/2}\right] \tag{1.51}
$$

which in other words means that the tropical Welschinger number $W^{(S,L),\delta}$ associated to the particular cases of a toric surface S and toric line bundle, is topological invariant depending only on the Chern numbers $(L^2, K_S^2, LK_S, c_2(S))$ of the pair (S,L).

CHAPTER 2

Refined Tropical Enumerative Geometry

We begin by recalling a few definitions from graph theory that will be useful in the subsequent sections. The standard terminologies used in graph theory and can be found in standard references such as [**BM76, Die10**]. Recall that an abstract graph $G = (V, E)$ is a pair of sets where V is a set of points called the vertices of G and $E \subset V \times V$ is the set of edges of G. E is a multiset meaning that elements can occur with multiplicity greater than one. G is said to be weighted if there exists a map $\rho : E \to \mathbb{Z}_{>0}$ called the weight function labeling its edges usually with the set of positive integers. G is said to be a directed graph if the elements of E are ordered pairs. Thus an edge $e = (u, v) \in E$ of a directed graph G has initial vertex u and terminal vertex v. If G is graph endowed with lengths on edges then one can define a metric d_G on G by setting $d_G(u, v)$ to be the length of the shortest path from u to v if such a path exists. If G is connected then the pair (V, d_G) is indeed a finite metric space.

2.1. Parameterized Tropical Curves

The references used in this section includes the foundational work of Mikhalkin [**Mik05**] as well as [**AB13, BG16, BIMS15**]. Other sources that may have been in one way or another used in this Chapter includes [**Gro11, IMS07**].

Let $\bar{\Gamma}$ be a weighted finite graph. This means that the set Γ^0 of vertices of $\bar{\Gamma}$ is finite, the set Γ^1 of edges of $\bar{\Gamma}$ is also finite and there is a map $w : \Gamma^1 \to \mathbb{Z}_{>0}$ associating to each edge $e \in \Gamma^1$ a positive integer $w(e)$ called its weight. Denote by Γ^0_∞ the subset of Γ^0 consisting of univalent vertices and let Γ^1_∞ denote the subset of edges of $\bar{\Gamma}$ adjacent to univalent vertices.

Definition 2.1. [**Mik05**, §2] An *abstract tropical curve* is a compact graph $\bar{\Gamma}$ without divalent vertices and isolated vertices such that $\Gamma = \bar{\Gamma} \backslash \Gamma^0_\infty$ is a metric graph whose compact edges are isometric to closed segments of $\mathbb{R}$ and the non-compact edges are isometric to $\mathbb{R}$ or to a rays in $\mathbb{R}$.

DEFINITION 2.2. [**Mik05**, §2] Let Δ be a convex lattice polytope in $\mathbb{Z}^n$. A *parameterized tropical curve* C of degree Δ is a pair $(\bar{\Gamma}, h)$ where $\bar{\Gamma}$ is an abstract tropical curve and $h : \Gamma \to \mathbb{R}^n$ is a continuous map such that

(a) every edge e of $h(\Gamma)$ lies in a unique affine line with rational slope, and is weighted with positive integer weight $w(e)$,
(b) and every vertex v of $h(\Gamma)$ is balanced meaning that

$$\sum_{e \in S_v} w(e)u(v,e) = \mathbf{0},$$

where S_v is the subset of all edges of $h(\Gamma)$ incident to v and $u(v,e)$ is the primitive integral vector starting from v and oriented 'away' in the direction of e,
(c) for each primitive vector $u \in \mathbb{Z}^n$, the total weight of the unbounded edges in the direction u equals the lattice length of an edge of $\partial\Delta$ with outer normal vector u (if there is no such edge we require the total weight to be zero).

DEFINITION 2.3. [**Cap13, BM07, Mik05**] Two parameterized tropical curves $C = (\bar{\Gamma}, h)$ and $C' = (\bar{\Gamma}', h')$ are said to have the same *combinatorial type* if there exists a homeomorphism $\phi : \Gamma \to \Gamma'$ such that for every edge e of Γ we have

$$w(h(e)) = w(h' \circ \phi(e)) \text{ and } \lambda(h(e)) = \lambda(h' \circ \phi(e))$$

where $\lambda(E)$ denote slope of the affine line containing the line segment E. The curves C and C' are said to be *isomorphic* if there exists an isomorphism of the underlying metric graphs $\psi : \Gamma \to \Gamma'$ such that $h = h' \circ \psi$. We shall consider parameterized tropical curves up to isomorphism.

EXAMPLE 2.4. Figure 2.1 depicts an example of a parameterized tropical curve C and its degree Δ. C has two edges of weight 2. Every edge of C that is not labeled has weight 1.

DEFINITION 2.5. [**Mik05**, §2] A tropical curve C is said to be irreducible if the underlying topological space of C has exactly one component. The genus of an

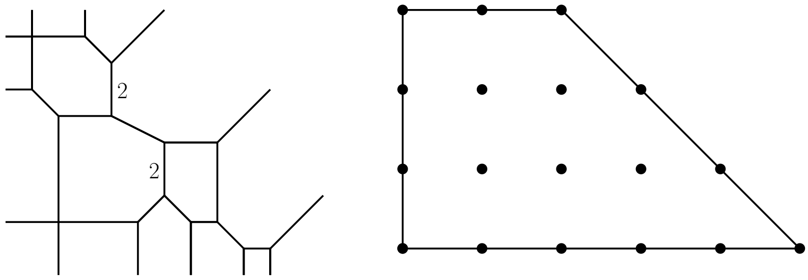


FIGURE 2.1. A tropical curve(left) and its degree(right).

irreducible curve $C = (\Gamma, h)$ is defined to be $g(C) := \dim H^1(h(\Gamma), \mathbb{R})$ i.e. the first Betti number of $h(\Gamma)$.

The dual subdivision $\Delta(C)$ corresponding to the tropical curve C is the unique subdivision of Δ whose 2-faces $\Delta(v)$ corresponds to vertices v of C in such a manner that each edge e of C incident to v is orthogonal to a unique edge in the boundary of $\Delta(v)$. A tropical curve C corresponds to a unique subdivision of Δ, however, the converse is not true. A subdivision of Δ determines the combinatorial type of the curve and not the curve itself. The problem of finding an algorithm to generate the subdivisions of a lattice polygon Δ has been studied extensively in geometric combinatorics [**IMTI02, Ram02**].

DEFINITION 2.6. [**Mik05**, §2] The tropical curve C of degree Δ is said to be *nodal* if its corresponding dual subdivision $\Delta(C)$ consists only of triangles and parallelograms.

EXAMPLE 2.7. The dual subdivision Δ_C of Δ corresponding to the tropical curve C in Example 2.4 is shown in Figure 2.2 and evidently, C is thus a nodal curve.

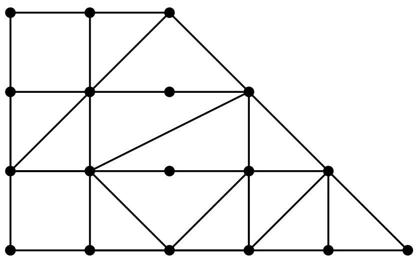


FIGURE 2.2. Dual subdivision Δ_C of Δ.

Definition 2.8. [**Mik05**, §4] A tropical curve $C = (\Gamma, h)$ is called simple if all the vertices of C are 3-valent, the self intersections of h are disjoint from the vertices, and the inverse image under h of self intersection points consists of exactly two points.

Definition 2.9. [**Mik05**, §4] The number of nodes of a nodal irreducible tropical curve C of degree Δ is $\delta(C) := \#int(\Delta) \cap \mathbb{Z}^2 - g(C)$ where $int(\Delta)$ denote the set of interior points of Δ. It is also equal to the number of parallelograms of the dual subdivision $\Delta(C)$ if C is a simple tropical curve. The number $\delta(C)$ is called the *cogenus* of the curve C.

Definition 2.10. [**BG16**, Def. 3.4] Let $C = (\Gamma, h)$ be a nodal curve with irreducible components $C_1, \ldots, C_r$ with $C_i = (\Gamma_i, h_i)$ having degree Δ_i and having δ_i nodes. Then the number of nodes of C is given by

$$\delta(C) = \sum_{i=1}^{r} \delta_i + \sum_{i<j} \mathcal{M}(\Delta_i, \Delta_j) \tag{2.1}$$

where $\mathcal{M}(\Delta_i, \Delta_j) := \frac{1}{2}(\text{Area}(\Delta_i + \Delta_j) - (\text{Area}(\Delta_i) + \text{Area}(\Delta_j)))$ and where $\text{Area}(\cdot)$ is defined to be twice the Euclidean area in $\mathbb{R}^2$.

2.2. Complex, Real and Refined Multiplicities

Definition 2.11. [**Mik05**, Def. 2.16] Let C be a simple tropical curve and v be a 3-valent vertex of C. Let w_1, w_2 and w_3 be the weights of the edges adjacent to v and let u_1, u_2, u_3 be the primitive integer vectors in the directions of the edges. The multiplicity of v is defined to be

$$\mu(v) := \text{Area}(\Delta_v) = w_1 w_2 |u_1 \times u_2|.$$

Note that by the balancing condition, this is also equal to $w_1 w_3 |u_1 \times u_3|$ and also equal to $w_2 w_3 |u_2 \times u_3|$. The *complex multiplicity* (also *Mikhalkin's multiplicity* [**Mik05**] or simply *multiplicity*) of a simple tropical curve C is defined to be

$$\mu_{\mathbb{C}}(C) := \prod_{v} \mu(v) \tag{2.2}$$

where the product is over all 3-valent vertices of C.

EXAMPLE 2.12. In the tropical curve C depicted in Example 2.4 above, it is easy to show that with the exception of the vertices incident to edges of weight 2, all the other 3-valent vertices have multiplicity $\mu(v) = 1$. Furthermore, each of the four vertices incident to the edges of weight 2 have multiplicity $\mu(v) = 2$. The tropical curve C therefore has multiplicity $\mu_{\mathbb{C}}(C) = 16$.

We also associate a tropical curve C with its *real multiplicity*. This multiplicity takes values in the set $\{0, -1, 1\}$ and is often called the sign of the curve C.

DEFINITION 2.13. [**BIMS15**, §4.2] Let C be a simple tropical curve and v be a trivalent vertex of C. The sign (also called the *mass*) of the vertex v is defined to be

$$m(v) = \begin{cases} (-1)^{(\mu(v)-1)/2} \text{ if } \mu(v) \text{ is odd and,} \\ 0 \text{ otherwise.} \end{cases} \tag{2.3}$$

The real multiplicity (also called the Welschinger sign) of C denoted by $\mu_{\mathbb{R}}(C)$ is defined to be the product of $m(v)$ over all 3-valent vertices of C.

EXAMPLE 2.14. Its clear from Definition 2.13 that once a tropical curve C has a vertex v such that $\mu(v)$ is even then $\mu_{\mathbb{R}}(C) = 0$. In particular the curve in Example 2.4 has real multiplicity zero since the curve has at least one vertex of multiplicity 2. Figure 2.3 depicts a tropical curve C with complex multiplicity $\mu_{\mathbb{C}}(C) = 3$ and real multiplicity $\mu_{\mathbb{R}}(C) = -1$.

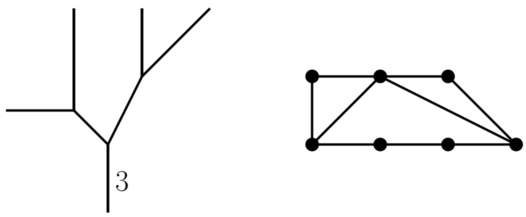


FIGURE 2.3. Tropical curve and corresponding subdivision.

So far we have introduce tropical curves and associated a tropical curve to its degree Δ, its cogenus $\delta(C)$, its complex multiplicity $\mu_{\mathbb{C}}(C)$ and its real multiplicity $\mu_{\mathbb{R}}(C)$. Our main aim is to use these values to show that the generating functions for the refined node polynomials introduced in Chapter 1, satisfy some interesting properties. For this purpose, we associate to a tropical curve a Laurent polynomial multiplicity called its *refined multiplicity*.

In Remark 1.19 we introduced for an integer n the Laurent polynomial $[n]_y$ (see also [**BG16**, §1]). This is given by

$$[n]_y := \frac{y^{n/2} - y^{-n/2}}{y^{1/2} - y^{-1/2}} = y^{(n-1)/2} + \cdots + y^{-(n-1)/2}.$$

For $y = 1$, then it can be shown that $[n]_y$ specializes to n.

DEFINITION 2.15. [**BG16**, Def. 3.5] Let C be a simple tropical curve. The *refined multiplicity* of a 3-valent vertex v is defined by $\mu(v, y) := [\mu(v)]_y$. The *refined multiplicity* of C is therefore defined to be

$$\mu(C, y) := \prod_{v} \mu(v, y) \tag{2.4}$$

with the product again running over all the 3-valent vertices of C. Note that in particular we have $\mu(C, 1) = \mu(C)$.

EXAMPLE 2.16. The refined multiplicity of the tropical curve C in Example 2.4 equals $\mu(C, y) = ([2]_y)^4 = (y^{1/2} + y^{-1/2})^4 = y^2 + 4y + 6 + 4y^{-1} + y^{-2}$, while the curve depicted in Figure 2.3 has refined multiplicity $\mu(C, y) = [3]_y = y + 1 + y^{-1}$.

REMARK 2.17. From the definition of $[n]_y$ it is easy to see that $\mu(C, y)$ will always be a Laurent polynomial symmetric with respect to $y \mapsto y^{-1}$. The refined multiplicity unifies the complex and the real multiplicity of a tropical curve i.e. $\mu(C, 1) = \mu_{\mathbb{C}}(C)$ and $\mu(C, -1) = \mu_{\mathbb{R}}(C)$ (see [**BG16, IM13, GS14**] for details).

2.3. Refined Severi Degrees on Toric Surfaces

We take a short detour to look at the refined Severi degrees on toric surfaces. A toric surface is a projective algebraic surface that contains $(\mathbb{C}^*)^2$ as a dense open subset. Classically, a complex algebraic curve $C \subset (\mathbb{C}^*)^2$ is defined to be the zero locus of a complex polynomial. Let

$$f(x, y) = \sum_{(i,j) \in \mathbb{Z}^2} a_{ij} x^i y^j$$

be a polynomial with $a_{ij} \in \mathbb{C}$ and finitely non-zero. The *Newton polygon* of f denoted by $\Delta(f)$ is the convex hull of the exponents vectors $\{(i, j) : a_{ij} \neq 0\}$. The

Newton polygon of polynomial generalizes the notion of degree of the polynomial. In particular, if $f, g \in \mathbb{C}[x, y]$ are polynomials then $\Delta(fg) = \Delta(f) + \Delta(g)$ i.e. the Minkowski sum of the respective polygons.

Fix a convex lattice polygon $\Delta \subset \mathbb{R}^2$ and $\delta \geq 0$. Let $C = \{(a, b) \in (\mathbb{C}^*)^2 : f(a, b) = 0\}$. We say that a curve $C \in (\mathbb{C}^*)^2$ is of degree Δ if $\Delta(f) = \Delta$. Let

$$k = |(\Delta \cap \mathbb{Z}^2)| - \delta - 1.$$

Definition 2.18. The *classical Severi degree* denoted by $N^{\Delta,\delta}$ is defined to be the number of reduced but possibly reducible δ-nodal curves $C \subset (\mathbb{C}^*)^2$ of degree Δ passing through a configuration $\omega = \{p_1, \ldots, p_k\} \subset (\mathbb{C}^*)^2$ of complex points in general position.

It is well understood [**CH98**] that $N^{\Delta,\delta}$ is a finite number and does not depend on the configuration of generic points. It is convenient to study $N^{\Delta,\delta}$ with the help of the toric surface associated to the polygon Δ. A convex lattice polygon Δ determines a toric surface S_Δ and an ample line bundle L_Δ on S_Δ. Any curve in $|L_\Delta|$ is the closure in S_Δ of the zero locus of a polynomial $f \in \mathbb{C}[x, y]$ whose Newton polygon is contained in Δ and therefore, $\dim |L_\Delta| = \#(\Delta \cap \mathbb{Z}^2)$.

A lot of the important data of the pair (S_Δ, L_Δ) is encoded in a lattice polygon Δ. For example let $|L_\Delta|$ be the complete linear system of curves on S_Δ that are linearly equivalent to L_Δ. Then we have that $\dim |L_\Delta| = \#(\Delta \cap \mathbb{Z}^2) - 1$ and furthermore, the arithmetic genus of a generic curve in $|L_\Delta|$ is equal to $\# int(\Delta \cap \mathbb{Z}^2)$. More details about this can be found in standard literature about toric varieties e.g. [**Ful93**].

Definition 2.19. The Severi degree for the pair (S_Δ, L_Δ) denoted by $N^{(S_\Delta,L_\Delta),\delta}$ is defined to be the number of δ-nodal curves in $|L_\Delta|$ passing through a configuration of $\#(\Delta \cap \mathbb{Z}^2) - \delta - 1$ general points in S_Δ.

We consider the tropical analogue of $N^{(S_\Delta,L_\Delta),\delta}$ i.e. the problem of counting the number of δ-nodal (simple) tropical curves of degree Δ, passing through a fixed configuration of *tropically generic* points on $\mathbb{R}^2$. Let

$$s = \#(\partial\Delta \cap \mathbb{Z}^2) \text{ and } l = \#(int(\Delta) \cap \mathbb{Z}^2). \tag{2.5}$$

Here, s is the number of unbounded edges of a tropical curve C of degree Δ with each edge counted with its weight and l is the genus of a *smooth tropical curve* of degree Δ. Denote by $x \leq s$ the number of unbounded edges of C with each curve counted "simply".

DEFINITION 2.20. [**Mik05**, Def. 4.7] Let $\omega = \{p_1, \ldots, p_r\}$ be a configuration of distinct points $\mathbb{R}^2$. Then ω is a tropically generic point configuration if for any tropical curve $C = (\Gamma, h)$ of degree Δ, genus g and with x ends such that $r \geq g+x-1$ and such that $\omega \subset h(\Gamma)$ satisfy the following

(1) $C = (\Gamma, h)$ is a simple tropical curve,
(2) $h^{-1}(p_1), \ldots, h^{-1}(p_r)$ are disjoint from the vertices of the graph Γ and
(3) $r = g + x - 1$.

By [**Mik05**, Prop. 4.11, Cor. 4.12], tropical generic point configurations form a dense set in $Sym^r(\mathbb{R}^2)$. Further, for any pair (Δ, g) with $g \leq l$, there exist only finitely many simple tropical curves of degree Δ and genus g passing trough a fixed configuration of tropically generic points on $\mathbb{R}^2$ [**Mik05**, Prop. 4.13]. A vertically stretched point configuration (will be useful later in §2.4) is defined to be the following.

DEFINITION 2.21. [**BG16**, Def. 3.6] A point configuration $\omega = \{(x_1, y_1), \ldots, (x_r, p_r)\} \subset \mathbb{R}^2$ is said to be vertically stretched with respect to Δ if for every curve of degree Δ, we have

$$\min_{i \neq j} |y_i - y_j| > \max_{i \neq j} |x_i - x_j| \cdot M(C)$$

where $M(C)$ is the maximal slope of an edge of C multiplied by the number of edges of C.

DEFINITION 2.22. The *tropical Severi degree* denoted by $N^{\Delta,\delta}_{\text{trop}}$ is defined to be the number of δ-nodal simple tropical curves of degree Δ passing through a configuration of $\#(\Delta \cap \mathbb{Z}^2) - \delta - 1$ tropically generic points in $\mathbb{R}^2$, with each curve counted with its complex multiplicity.

THEOREM 2.23 (Mikhalkin's correspondence theorem). [**Mik05**, Thm. 1] *For any integer $\delta \geq 0$ and $\Delta \subset \mathbb{R}^2$ a convex lattice polygon then*

$$N^{(S_\Delta, L_\Delta),\delta} = N^{\Delta,\delta}_{\text{trop}}, \tag{2.6}$$

and the number $N^{\Delta,\delta}_{\text{trop}}$ does not depend on the generic point configuration.

Theorem 2.23 above provides us with a recipe of obtaining the number of complex algebraic curves on toric surfaces satisfying prescribed conditions by studying the corresponding problem on the tropical side. We also consider the tropical version for the computation of the Welschinger invariants introduced in §1.5 above.

DEFINITION 2.24. [**Mik05**, Def. 7.11] The *tropical Welschinger number* denoted by $W^{\Delta,\delta}_{\text{trop}}(\omega)$ is defined to be the number of δ-nodal simple tropical curves of degree Δ passing through a configuration ω of $\#(\Delta \cap \mathbb{Z}^2) - \delta - 1$ tropically generic points in $\mathbb{R}^2$ counted with the real multiplicity.

Welschinger [**Wel03, Wel05**], (also [**Mik05**, Thm. 5]) showed that if S_Δ is smooth and rational then the signed count of real rational curves passing trough sufficiently many tropical general points is an invariant independent of the point configurations. In general, Welschinger numbers depend on the real point configurations on S_Δ. The tropical Welschinger numbers $W^{\Delta,\delta}_{\text{trop}}(\omega)$ are independent of the tropical point configurations. In [**Mik05**, Thm 3] it is shown that for every tropically generic point configuration ω there exists a configuration $\mathcal{R}$ of points on S_Δ (called subtropical point configurations) such that the Welschinger number $W^{(S_\Delta,L_\Delta),\delta}(\mathcal{R}) = W^{\Delta,\delta}_{\text{trop}}(\omega)$.

We shall assume that a subtropical point configuration has been chosen and write $W^{\Delta,\delta}$ (respectively $W^{(S,L),\delta}$) for $W^{\Delta,\delta}_{\text{trop}}(\omega)$ (respectively $W^{(S,L),\delta}(\mathcal{R})$). We now consider the refined analogues of $N^{\Delta,\delta}$ and $W^{\Delta,\delta}$. Itenberg and Mikhalkin [**IM13**, Theorem 1] showed that the refined multiplicity of tropical curves in independent of the choice of generic point configuration. Consequently, the refined tropical Severi degree defined below is also an invariant independent of the point configuration.

DEFINITION 2.25. The *refined tropical Severi degree* denoted by $N^{\Delta,\delta}_{\text{trop}}(y)$ is defined to be the number of δ-nodal simple tropical curves of degree Δ passing through a configuration of $\#(\Delta \cap \mathbb{Z}^2) - \delta - 1$ tropically generic points in $\mathbb{R}^2$ counted with the refined multiplicity.

Göttsche and Shende [**GS14**] introduced the refined Severi degrees $N^{(S,L),\delta}(y)$ for $\mathbb{P}^2$ and the rational ruled surface. $N^{(S,L),\delta}(y)$ is defined by the modified version of Caporaso-Harris recursion discussed in §1.4.3. Block and Göttsche [**BG16**, Thm 7.5] determined a Caporaso-Harris type recursion formula for the weighted projective space $\mathbb{P}(1,1,m)$ and proved the following theorem.

THEOREM 2.26. [**BG16**, Thm. 1.1] *Let S_Δ be $\mathbb{P}^2$, a rational ruled surface Σ_m or the weighted projective space $\mathbb{P}(1,1,m)$. Then*

$$N^{(S_\Delta,L_\Delta),\delta}(y) = N^{\Delta,\delta}_{trop}(y).$$

By Remark 2.17 it follows that in the case S_Δ is $\mathbb{P}^2$, a rational ruled surface Σ_m or the weighted projective space $\mathbb{P}(1,1,m)$ then Theorem 2.26 specializes at $y=1$ to the Mikhalkin's correspondence (Theorem 2.23). The immediate implication of Theorem 2.26 is that we can use tropical methods to study the refined Severi degrees and their generating functions. On the other hand, refined tropical Severi degrees can be studied by looking at the combinatorics of *floor diagrams*. These are purely combinatorial graphs introduced by Brugallé and Mikhalkin [**BM07, BM09**] that are used to encode some of the most important aspects of the tropical curve.

2.4. Refined Tropical Severi Degrees via Floor Diagrams

We review the floor diagrams associated to curves on toric surfaces which are defined by *h-transverse* lattice polygons. Recall that a lattice polygon Δ is a polygon in $\mathbb{R}^2$ whose vertices are points of the integral lattice. The interior and the boundary of a lattice polygon are denoted respectively by $int(\Delta)$ and $\partial\Delta$. The lattice length of a lattice segment e in $\mathbb{R}^2$ is defined to be $l(e) := \#(\mathbb{Z}^2 \cap e) - 1$. Let Δ be a convex lattice polygon. Its left and its right boundaries denoted by $\partial_l\Delta$ and respectively by $\partial_r\Delta$ are defined to be

$$\partial_l\Delta := \{p \in \partial\Delta | \forall t > 0, \quad p + (-t,0) \notin \Delta\}, \tag{2.7}$$

$$\partial_r\Delta := \{p \in \partial\Delta | \forall t > 0, \quad p + (t,0) \notin \Delta\}. \tag{2.8}$$

DEFINITION 2.27. If v is a vertex of a lattice polygon $\Delta \in \mathbb{R}^2$ then the determinant $\det(v)$ of v is defined to be $\det|w_1, w_2|$ where w_1 and w_2 are primitive integer normal vectors to the edges adjacent to v.

DEFINITION 2.28. [**AB13**, §1],[**BM09**, §2]. A lattice polygon is said to be *h-transverse* if every edge of Δ has slope $0, \infty$ or $1/k$ for some integer k. Alternatively, Δ is h-transverse if any primitive vector parallel to an edge in $\partial_l\Delta$ or $\partial_r\Delta$ is of the form $(\alpha, -1)$ with $\alpha \in \mathbb{Z}$. The lattice polygon Δ is said to be strongly h-transverse if either there is a non-zero horizontal edge at the top of Δ or the vertex v at top of Δ has $\det(v) \in \{1, 2\}$, and the same holds for the bottom of Δ.

Let d^t(respectively d^b) be the length of the horizontal edge at the top (resp. at the bottom) of Δ is it exists, otherwise set $d^t = 0$(resp. $d^b = 0$) if no such edge exists. The left directions denoted by $\boldsymbol{l}$(resp. right directions denoted by $\boldsymbol{r}$) of Δ is defined to be the unordered list that consists of elements α repeated $l(e)$ times for an edge $e = \pm l(e)(\alpha, -1)$ of $\partial_l\Delta$(resp. $\partial_r\Delta$).

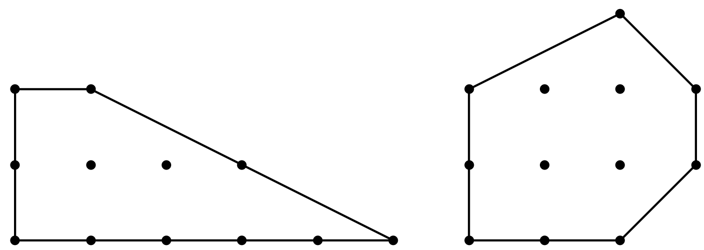
There is a 1-1 correspondence [**BM09**, §2] between quadruples $(\boldsymbol{l}, \boldsymbol{r}, d^t, d^b)$ and h-transverse polygons Δ considered up to translations. Furthermore,

$$\#d^l = \#d^r = \#(\partial_l\Delta \cap \mathbb{Z}^2) - 1 = \#(\partial_r\Delta \cap \mathbb{Z}^2) - 1 \tag{2.9}$$

and also

$$2\#(\partial_l\Delta \cap \mathbb{Z}^2) + d^t + d^b = \#(\partial\Delta \cap \mathbb{Z}^2). \tag{2.10}$$

Example 2.29. Some h-transverse polygons are depicted in Figure 2.4 below



(a) $\boldsymbol{l} = \{0, 0\}$
$\boldsymbol{r} = \{2, 2\}$
$d^t = 1$
$d^b = 5$

(b) $\boldsymbol{l} = \{-2, 0, 0\}$
$\boldsymbol{r} = \{1, 0, -1\}$
$d^t = 0$
$d^b = 2$

Figure 2.4. Some h-transverse polygons and their respective left and right directions.

We are primarily interested in floor diagram associated to curves on $\mathbb{P}^2$, rational ruled surfaces and the weighted projective surfaces $\mathbb{P}(1, 1, m)$. The corresponding lattice polygons are of type $\Delta_{c,m,d}$ where $\Delta_{c,m,d} = \{(x, y) \in (\mathbb{R}_{\geq 0})^2 : y \leq d; x + my \leq md + c\}$ for $d, m, c \geq 0$. In particular if:

(1) $d \geq 0$, $m = 1$, $c = 0$ then $S(\Delta_{0,1,d}) = \mathbb{P}^2$, $L(\Delta_{0,1,d}) = dH$, with H the hyperplane bundle on $\mathbb{P}^2$;

(2) $d \geq 0$, $m \geq 1$, $c = 0$ then $S(\Delta_{0,m,d}) = \mathbb{P}(1,1,m)$, $L(\Delta_{0,m,d}) = dH$, with H the hyperplane bundle on $\mathbb{P}(1,1,m)$ with self intersection m;

(3) $d \geq 0$, $m \geq 0, c \geq 0$ then $S(\Delta_{c,m,d})$ is the m-th rational ruled surface $\Sigma_m := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(m))$. Let F be the class of the fibre of the ruling and let E be the class of a section with $E^2 = -m$. We denote $H := E + mF$. Then $L(\Delta_{c,m,d}) = cF + dH$.

We shall focus mainly on the lattice polygons listed above even though the methods that shall be discussed can be adapted to work for any h-transverse lattice polygons. The particular polygons described in (1) – (3) above are listed in Figure 2.5 below.

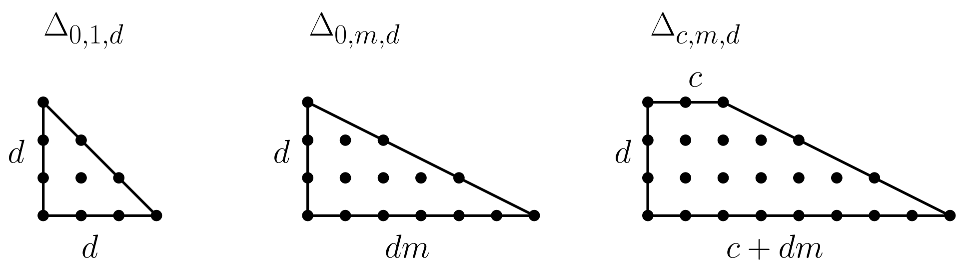


FIGURE 2.5. Lattice polygons for $\mathbb{P}^2, \mathbb{P}(1,1,m)$ and Σ_m.

DEFINITION 2.30. [**BG16**, §5] Fix c, m, d and write $\Delta := \Delta_{c,m,d}$. A Δ-*floor diagram* is the data of a weighted, directed acyclic graph $\mathcal{D} = (V, E)$ and a sequence $(s_1, \ldots, s_d)$ of nonnegative integers satisfying the following conditions.

(1) $\mathcal{D}$ is a graph on a vertex set $V = \{1, \ldots, d\}$ and may have multiple edges between a pair of vertices. Edges $e \in E$ of $\mathcal{D}$ have positive integer weights and are oriented $i \to j$ if $i < j$. The graph $\mathcal{D}$ has no loops.

(2) The sequence $(s_1, \ldots, s_d)$ satisfies $s_1 + \cdots + s_d = c$.

(3) (Divergence Condition) For each vertex j of $\mathcal{D}$, we have

$$\operatorname{div}(j) \stackrel{\text{def}}{=} \sum_{\substack{e \in E \\ j \xrightarrow{e} k}} w(e) - \sum_{\substack{e \in E \\ i \xrightarrow{e} j}} w(e) \leq m + s_j.$$

Condition (3) above says that at every vertex of $\mathcal{D}$ the total weight of the outgoing edges is larger by at most $m + s_j$ than the total weight of the incoming edges. When $\Delta = \Delta_{c,m,d}$ is clear from the context we simply say that $\mathcal{D}$ is a floor diagram.

The *arithmetic genus* $g_a(\mathcal{D})$ of a Δ-floor diagram is defined to be the number of interior lattice points of Δ. The diagram $\mathcal{D}$ is said to be *connected* if the underlying graph is connected as a topological space. If $\mathcal{D}$ is connected then its *genus* $g(\mathcal{D})$ is defined to be the first Betti number of the underlying graph and thus its *cogenus* is defined to be

$$\delta(\mathcal{D}) := g_a(\mathcal{D}) - g(\mathcal{D}) = \#\mathrm{int}(\Delta \cap \mathbb{Z}^2) - \#E + d - 1. \tag{2.11}$$

Suppose that $\mathcal{D}$ is not connected and that $\mathcal{D}_1, \ldots, \mathcal{D}_r$ are the connected components of $\mathcal{D}$. Then each $\mathcal{D}_i$ is a connected Δ_i-floor diagram for some lattice polygon Δ_i. Then the Minkowskii sum of the lattice polygons Δ_i satisfies $\Delta_1 + \cdots + \Delta_r = \Delta$. Let $\delta_1, \ldots, \delta_r$ be the cogenera of the connected components then

$$\delta(\mathcal{D}) = \sum_{i=1}^{r} \delta_i + \sum_{i<j} \mathcal{M}(\Delta_i, \Delta_j)$$

where as in the case of tropical curves, $\mathcal{M}(\Delta_i, \Delta_j) := \frac{1}{2}(\mathrm{Area}(\Delta_i+\Delta_j) - (\mathrm{Area}(\Delta_i) + \mathrm{Area}(\Delta_j)))$.

Example 2.31. Let $(c, m, d) = (2, 1, 3)$ i.e. $\Delta = \Delta_{2,1,3}$ and let $(s_1, s_2, s_3) = (1, 1, 0)$. The graph in Figure 2.6 satisfies conditions (1)–(3) in Definition 2.30 above and is therefore a Δ-floor diagram.

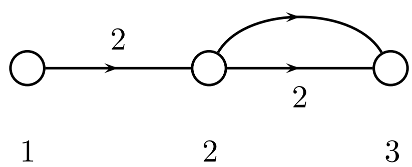


Figure 2.6. An example of a floor diagram.

There is an almost canonical correspondence between tropical curves of degree Δ and the Δ-floor diagrams. Let C be a tropical curve of degree Δ. Define an *elevator* of C to any vertical edge of C i.e. any edge parallel to the vector $(0, 1)$.

The multiplicity of an elevator is inherited from the multiplicity of that edge in the tropical curve C. A *floor* of C is defined to be any connected component of C that does not contain an elevator. We create a graph $\mathcal{D}$ by the following steps

Step 1: Contract each floor of C into a point creating a vertex of a graph denoted $\widetilde{\mathcal{D}}$. The directed edges of $\widetilde{\mathcal{D}}$ are the elevators of C oriented to point "downwards" i.e. in the direction of the vector $(0, -1)$.

Step 2: Let $\mathcal{D}$ be the graph obtained by removing the univalent edges of $\widetilde{\mathcal{D}}$. The univalent edges of $\widetilde{\mathcal{D}}$ corresponds to the non-compact elevators of the tropical curve C.

This procedure is illustrated in Figure 2.7 below. The diagram to the left is a tropical curve C of degree $\Delta = \Delta_{(2,1,3)}$. The elevators of C are the vertical edges shown in dark/bold edges while the floors are illustrated in light/gray connected components of C. In the middle is the intermediate directed graph $\widetilde{\mathcal{D}}$ obtained by contracting each floor to a point. To the right is the directed graph $\mathcal{D}$ obtained by deleting the univalent edges of $\widetilde{\mathcal{D}}$.

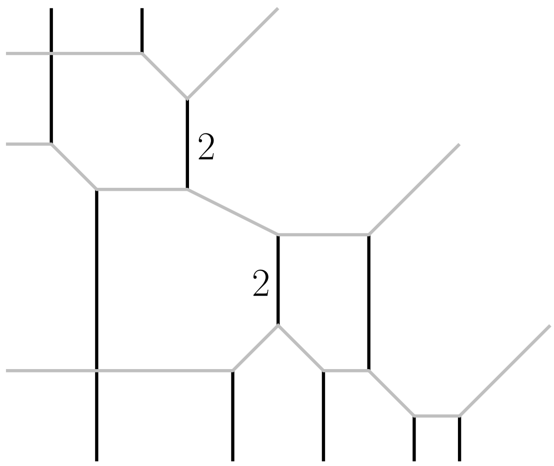


FIGURE 2.7. Obtaining a floor diagram from a tropical curve.

The h-transversality condition of the lattice polygon Δ guarantees that for vertically stretched generic point configuration ω then the tropical curves interpolating through the points must have exactly one point of ω on each elevator and exactly one point on each floor. For such a tropical curve, the diagram $\mathcal{D}$ obtained in the two step procedure above is a floor diagram in the sense of Definition 2.30. For space preserving reasons, floor diagram are usually drawn horizontally i.e. the diagram $\mathcal{D}$ obtained as above is rotated 90° counterclockwise.

Remark 2.32. Let $\Delta = \Delta_{(c,m,d)}$ be a lattice polygon and C be a tropical curve through a vertically stretched point configuration. Then the intermediate diagram $\widetilde{\mathcal{D}}$ obtained in the first step of the above procedure will have: d vertices; c univalent edges oriented *"inwards"* to the graph $\widetilde{\mathcal{D}}$ and $c+dm$ univalent edges oriented *"away"* from the graph $\widetilde{\mathcal{D}}$. Furthermore, there is a canonical ordering of the vertices of $\widetilde{\mathcal{D}}$ which is the "*top to bottom*" ordering. In this ordering every vertex j of $\widetilde{D}$ will have divergence

$$\operatorname{div}(j) := \sum_{\substack{e \in E \\ j \xrightarrow{e} k}} w(e) - \sum_{\substack{e \in E \\ i \xrightarrow{e} j}} w(e) = m + s_j.$$

Consequently, upon removal of the univalent vertices of $\widetilde{\mathcal{D}}$ we obtain a graph $\mathcal{D}$ satisfying condition (3) of Definition 2.30.

The floor diagram $\mathcal{D}$ associated to a tropical curve C passing through a vertically stretched point configuration encodes all the necessary geometric informations of C. Consequently, we can use this correspondence to create a recipe for studying tropical enumerative problems by reducing them to combinatorial problems on floor diagrams. Like in the case of tropical curves, we associate floor diagrams with refined multiplicities.

Definition 2.33. [**BG16**, Def. 5.2] Let $\mathcal{D}$ be a floor diagram. The *refined multiplicity* of $\mathcal{D}$ is defined to be

$$\mu(\mathcal{D}, y) := \prod_e \left([w(e)]_y\right)^2.$$

Example 2.34. We will use the floor diagram $\mathcal{D}$ in Figure 2.30 above(it should be clear that this is also the same as the diagram obtained in Figure 2.7 above). Note also that edges of weight 1 contribute a factor of 1 to the refined multiplicity. The refined multiplicity of D is thus

$$\mu(\mathcal{D}, y) = ([2]_y)^2 \cdot ([2]_y)^2 = y^2 + 4y + 6 + 4y^{-1} + y^{-2}.$$

This is precisely the refined multiplicity of the associated tropical curve C (Example 2.16).

2.5. Marked Floor Diagrams and Combinatorial Correspondence

To enumerate tropical curves via floor diagrams we need to count certain markings on these diagrams. Note that distinct tropical curves interpolating through vertically stretched point configurations may correspond to a unique floor diagram. Thus, counting the markings of floor diagrams can be interpreted as counting the number of the corresponding tropical curves. We will use the floor diagram in Figure 2.6 to illustrate the steps in the definition of the markings.

Definition 2.35. [**BG16**, Def. 5.4]. Let $\mathcal{D}$ be a Δ-floor diagram and $(s_1, \dots, s_d)$ be a sequence of nonnegative integers satisfying condition (2) of Definition 2.30. A *marking* of $\mathcal{D}$ is defined by the following four step procedure.

Step 1: For each vertex j of $\mathcal{D}$ create s_j new indistinguishable vertices and connect them to j by new unweighted edges directed towards j.

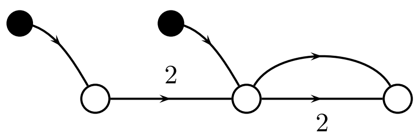


Figure 2.8. The floor diagram after Step 1.

Step 2: For each vertex j of $\mathcal{D}$ create $m + s_j - \mathrm{div}(j)$ new indistinguishable vertices and connect them to j with new edges directed away from j.

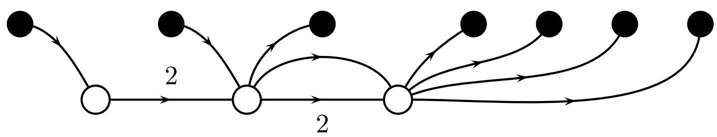


Figure 2.9. The result after applying Step 2.

Step 3: Subdivide each edge of the original floor diagram $\mathcal{D}$ into two directed edges by introducing a new vertex in the middle of each edge. The new edges inherit their weights and orientations form the original edges. The resulting graph is denoted by $\widetilde{\mathcal{D}}$.

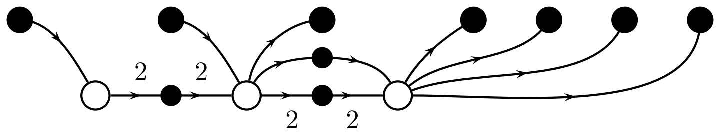


FIGURE 2.10. The result after applying Step 3.

Step 4: Linearly order the vertices of $\widetilde{\mathcal{D}}$ extending the order of the vertices of the original floor diagram $\mathcal{D}$ such that each edge is directed from a smaller to a larger vertex.

The extended graph $\widetilde{\mathcal{D}}$ together with the linear order on its vertices is called a *marked floor diagram* or a marking of the floor diagram $\mathcal{D}$. Two markings $\widetilde{\mathcal{D}_1}$ and $\widetilde{\mathcal{D}_2}$ of a floor diagram $\mathcal{D}$ are said to be equivalent if there exists an automorphism of weighted graphs which preserves the vertices of $\mathcal{D}$ and maps $\widetilde{\mathcal{D}_1}$ to $\widetilde{\mathcal{D}_2}$. The number of markings $\nu(\mathcal{D})$ of the floor diagram is defined to be the number of inequivalent markings of $\mathcal{D}$.

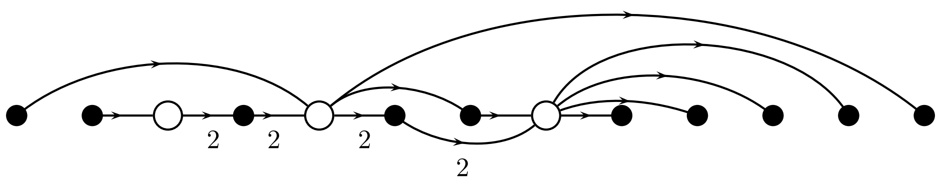


FIGURE 2.11. A linear order of the vertices of the diagram in Figure 2.10.

DEFINITION 2.36. [**BG16**, Def. 5.6] Fix $\delta \geq 0$ and let Δ be a h-transverse lattice polygon. The *combinatorial refined Severi degree* is defined to be the Laurent polynomial

$$N^{\Delta,\delta}_{\text{comb}}(y) := \sum_{\mathcal{D}} \mu(\mathcal{D}, y)\nu(\mathcal{D}) \tag{2.12}$$

where the sum is over all Δ-floor diagrams $\mathcal{D}$ of cogenus δ.

The combinatorial refined Severi degrees agree with the tropical refined Severi degrees (Definition 2.22) as stated in the Theorem 2.37 below. If S_Δ is smooth and L_Δ is sufficiently ample then $N^{\Delta,\delta}_{\text{comb}}(y)$ also coincide conjecturally with the refined invariants of Göttsche and Shende [**GS14**].

THEOREM 2.37 (Combinatorial correspondence). [**BG16**, Thm. 5.7] *Fix $\delta \geq 0$ and let Δ be a h-transverse lattice polygon. Then*

$$N^{\Delta,\delta}_{\mathrm{comb}}(y) = N^{\Delta,\delta}(y).$$

Theorem 2.37 is significant in the sense that it avails a recipe for using purely combinatorial graphs to study the refined counts of complex algebraic curves on toric surfaces. For the remainder of this section we show that $N^{\Delta,\delta}_{\mathrm{comb}}(y)$ is given by a polynomial in the parameters (c, m, d) defining Δ. To do this we use *templates* introduced by Fomin and Mikhalkin [**FM10**]. The templates are just floor diagrams satisfying additional "irreducibility" conditions.

DEFINITION 2.38. A template Γ is a weighted, directed graph on a vertex set $\{0, \dots, \ell\} \subset \mathbb{Z}$ satisfying the following conditions:

(1) multiple edges between a pair vertices i, j are allowed but no loops are allowed. For an edge $i \xrightarrow{e} j$ in Γ we must have $i < j$ and the weight $w(e)$ of an edge of Γ is a positive integer,
(2) Γ does not have *short edges*. These are edges of weight 1 and connecting two consecutive vertices i.e. edges of the form $i \xrightarrow{e} i + 1$,
(3) for each vertex j such that $1 \leq j < \ell$ there is an edges $i \xrightarrow{e} k$ with $i < j \leq k$.

In other words, a template can be said to be the connected component of what remains from a marked floor diagram upon removal of the short edges. We illustrate the process of obtaining templates by a slight modification the four step procedure in Definition 2.35. Let $\Delta = \Delta_{(c,m,d)}$ and $\mathcal{D}$ be a Δ-floor diagram. Recall that $\mathcal{D}$ is a graph on the vertex set $\{1, \dots, d\}$ (Definition 2.30).

PROCEDURE 2.39. The steps of Definition 2.35 is modified as follows:

Step 1: create a new vertex 0 and connect it to each vertex $j \geq 1$ by s_j new indistinguishable edges directed towards j;
Step 2: create a new vertex $d+1$ and connect it to each vertex $j : 1 \leq j \leq d$ with $m + s_j - \mathrm{div}(j)$ new indistinguishable edges directed away from j;
Step 3: remove short edges from the resulting graph, one obtains a graph which in general is a union of *shifted* templates.

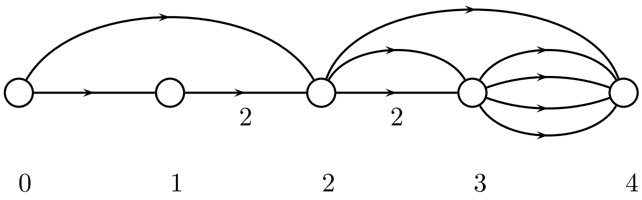


FIGURE 2.12. Graph obtained after applying of Step 1 and Step 2 of Procedure 2.39.

DEFINITION 2.40. Let Γ be a template. A shifted template denoted $\Gamma_{(k)}$ is the graph obtained from Γ by shifting all the edges of Γ by $k \in \mathbb{Z}$.

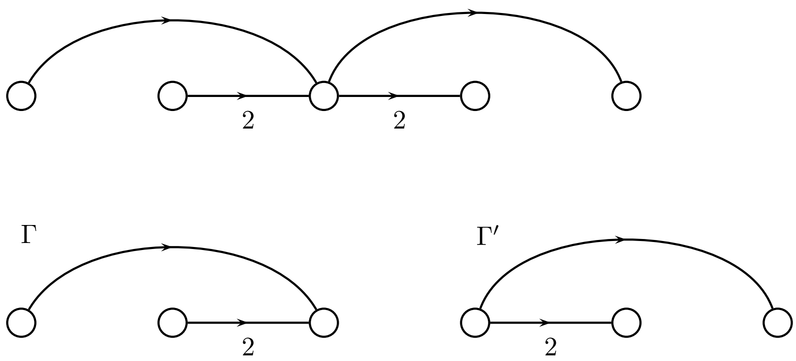


FIGURE 2.13. The resulting graph upon the removal of short edges from Figure 2.12 is a union of two shifted templates.

Each template Γ is associated with a number of important data. The length of a template $\ell = \ell(\Gamma)$ is defined to be the number of its vertices minus 1. Its cogenus is defined to be the number

$$\delta(\Gamma) := \sum_{i \xrightarrow{e} j} (j-i)w(e) - 1$$

and its refined multiplicity [**BG16**, Def. 5.9] is defined to be the Laurent polynomial

$$\mu(\Gamma, y) := \prod_{e} \left([w(e)]_y\right)^2$$

where the product is over all the edges of the template. Each of the templates Γ, Γ' in Figure 2.13 has cogenus 2 and refined multiplicity $([2]_y)^2 = y + 2 + y^{-1}$. We set

$$\varepsilon_0(\Gamma) = \begin{cases} 1 & \text{if all edges of } \Gamma \text{ starting at } 0 \text{ have weight } 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\varepsilon_1(\Gamma) = \begin{cases} 1 & \text{if all edges of } \Gamma \text{ arriving at } \ell \text{ have weight } 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $1 \le j \le \ell(\Gamma)$, let $\lambda_j = \lambda_j(\Gamma)$ denote the sum of weights of edges $i \xrightarrow{e} k$ with $i < j \le k$. By definition of a template, it follows that $\lambda_j > 0$ for all $1 \le j \le \ell(\Gamma)$. Denote $\lambda(G) = (\lambda_1, \ldots, \lambda_\ell)$. For a rational number a denote by $\lceil a \rceil$ the smallest integer bigger or equal to a and define

$$k_{\min}(\Gamma) := \max\left(1, \max_{1\le j\le \ell}\left\lceil \frac{\lambda_j - c - m(j-1)}{m} \right\rceil\right).$$

This makes $k_{\min}(\Gamma)$ the smallest positive integer k such that Γ can appear in a $\Delta_{c,m,d}$-floor diagram on $\{1, 2, \ldots\}$ with the left most vertex k.

Remark 2.41. Denote by $\widetilde{\mathcal{D}}$ the graph obtained after applying Procedure 2.39 above(see Figure 2.12). Deleting short edges from $\widetilde{\mathcal{D}}$ one obtains a uniquely defined collection of non-overlapping templates. Let $\Gamma_1, \ldots, \Gamma_r$ be these templates listed in the order of appearance from left to right. Denote by k_i the leftmost vertex of Γ_i (k_i is called the shift or the offset of Γ_i). Then the k_i satisfy the following inequalities

$$k_i + \ell(\Gamma_i) \le k_{i+1} \text{ for } 1 \le i \le r-1 \text{ and} \tag{2.13}$$

$$k_r + \ell(\Gamma_r) \le d + \varepsilon_1(\Gamma_r). \tag{2.14}$$

It is not hard to see that in $\widetilde{\mathcal{D}}$, for any vertex $v : 1 \le v \le d+1$, the total weight of all edges $u \xrightarrow{e} w$ with $u < v \le w$ is precisely equal to $m(v-1)+c$. Let s_v be the short edges connecting $v-1$ to v then $m(v-1)+c-s_v$ is the total weight of long edges appearing in one of the templates $\Gamma_1, \ldots, \Gamma_r$. If $v = k_i + j$ so that v belongs to a template Γ_i(as vertex j) then we have $\lambda_j(\Gamma_i) = m(v-1)+c-s_v = m(k_i+j-1)+c-s_v$. In other words $mk_i \ge \lambda_j(\Gamma_i) - c - m(j-1)$ which implies that

$$k_i \ge k_{\min}(\Gamma_i) \text{ for } 1 \le i \le r. \tag{2.15}$$

Conversely, given a sequence of isomorphism types of templates $\Gamma_1, \ldots, \Gamma_r$ and an increasing sequence of nonnegative integers $k_1, \ldots, k_r$ satisfying (2.13) – (2.15), there is a unique floor diagram $\mathcal{D}$ whose modification $\widetilde{\mathcal{D}}$ is obtained by placing each Γ_i with an offset k_i and adding sufficiently enough short edges.

By Remark 2.41 above, it follows that:

LEMMA 2.42. *There exists a 1-1 correspondence between marked floor diagrams and sequences of pairs* $(\Gamma_1, k_1), \ldots, (\Gamma_r, k_r)$ *of templates and nonnegative integers* k_i *satisfying* (2.13) – (2.15) *above.*

The above lemma has been discussed in detail in [**BG16**, 5.3, 5.4],[**AB13**, §3.2] and in the [**FM10**, 5.6]. Furthermore, it is easy to show that this correspondence preserves the cogenus i.e

$$\delta(\mathcal{D}) = \sum_{i=1}^{r} \delta(\Gamma_i).$$

It follows therefore that the number of markings of $\mathcal{D}$ can be obtained by counting the number of markings of the corresponding templates Γ_i.

Let Γ be a template and $k \geq 1$ be an integer. Denote by $\mathrm{ext}_{(c,m,k)}(\Gamma)$ the graph obtained by first adding

$$c + (k + j - 1)m - \lambda_j(\Gamma)$$

short edges connecting $j-1$ to j for $1 \leq j \leq \ell(\Gamma)$, then inserting an extra vertex in the middle of every edge of the resulting graph. Let $P_\Gamma(c, m, k)$ be the number of inequivalent linear extensions of the vertex poset of the graph $\mathrm{ext}_{(c,m,k)}(\Gamma)$ extending the order of the vertices of Γ. For any floor diagram $\mathcal{D}$ and and sequences of pairs $(\Gamma_1, k_1), \ldots, (\Gamma_r, k_r)$ of templates and nonnegative integers k_i satisfying (2.13) – (2.15) then the above discussion shows that

$$\nu(\mathcal{D}) = \prod_{i=1}^{r} P_{\Gamma_i}(c, m, k_i).$$

Consequently, the above discussion leads to the following proposition.

PROPOSITION 2.43. [**BG16**, Prop. 5.11] *Let*

(1) $S = \mathbb{P}^2$, $\delta \geq 1$ *and* $d \geq 1$*; or*
(2) $S = \mathbb{P}(1, 1, m)$ *and* $m, d \geq 1$ *and* $m \geq 2\delta$*; or*
(3) $S = \Sigma_m, \delta \geq 1$ *and* $m, c, d \geq 1$ *and* $m + c \geq 2\delta$

then

$$N^{\Delta,\delta}_{\mathrm{comp}}(y) = \sum_{\Gamma_1,\ldots,\Gamma_r} \left(\prod_{i=1}^{r} \mu(\Gamma_i, y) \sum_{k_1,\ldots,k_r} \left(\prod_{i=1}^{r} P_{\Gamma_i}(c, m, k_i) \right) \right) \tag{2.16}$$

where the first sum is over all tuples $(\Gamma_1, \dots, \Gamma_r)$ *satisfying* $\delta(\Gamma_1) + \cdots + \delta(\Gamma_r) = \delta$ *and the second sum is over all sequences* $k_1, \dots, k_r$ *of nonnegative integers satisfying* (2.13) – (2.15) *above.*

Proposition 2.43 coupled with Theorem 2.37 further reduces the problem of computing the refined Severi degree on $\mathbb{P}^2, \mathbb{P}(1,1,m)$ and Σ_m to an analysis of the combinatorics on the templates. Using the templates, Mikhalkin [**Mik05**, Thm. 5.1] showed that there exists a polynomial $N_\delta(d)$ such that whenever d is sufficiently large compared to δ then $N^{d,\delta} = N_\delta(d)$. The templates are also used by Ardila and Block [**AB13**, Thm. 1.2] to generalize the result of Mikhalkin to a larger class of toric surfaces, specifically, toric surfaces defined by h-transverse lattice polygons. In the refined scenario, Block and Göttsche [**BG16**, Thm 4.2] used the combinatorics of the templates to prove the following theorem about the polynomiality of the refined Severi degrees.

THEOREM 2.44. [**BG16**, Thm. 4.2]. *For fixed* $\delta \geq 1$ *we have the following.*

(1) $\mathbb{P}^2$*: There is a polynomial* $N_\delta(d; y) \in \mathbb{Q}[y^{\pm 1}][d]$ *of degree at most* 2δ *in* d *such that for* $d \geq \delta$*,*

$$N_\delta(d; y) = N^{d,\delta}_{\text{trop}}(y).$$

(2) Σ_m*: There is a polynomial* $N_\delta(c, d, m; y) \in \mathbb{Q}[y^{\pm 1}][c, m, d]$ *of degree at most* δ *in* c, m *and degree* 2δ *in* d *such that for* $c + m \geq 2\delta$ *and* $d \geq \delta$

$$N_\delta(c, d, m; y) = N^{(\Sigma_m, cF+dH),\delta}_{\text{trop}}(y).$$

(3) $\mathbb{P}(1,1,m)$*: There is a polynomial* $N_\delta(d, m; y) \in \mathbb{Q}[y^{\pm 1}][m, d]$ *of degree* δ *in* m *and degree at most* 2δ *in* d *such that for* $m \geq 2\delta$ *and* $d \geq \delta$

$$N_\delta(d, m; y) = N^{(\mathbb{P}(1,1,m),dH),\delta}_{\text{trop}}(y).$$

CHAPTER 3

Multiplicativity of the Generating Functions

The purpose of this chapter is to specialize the Conjecture 1.27 and Conjecture 1.28 to a large family of possibly singular toric surfaces. Let $\Delta = \Delta(c, m, d)$ be a lattice polygon such that the toric surface and the line bundle defined by Δ is $(\mathbb{P}^2, dH)$ or $(\mathbb{P}(1,1,m), dH)$ or $(\Sigma_m, cF + dH)$. For $\delta \geq 0$, denote by $N_\delta(c, m, d; y)$ the polynomial in Theorem 2.44 above and call them the refined node polynomials. By Theorem 2.37 and Definition 2.36 we have

$$N_\delta(c, m, d; y) = \sum_{\mathcal{D}} \mu(\mathcal{D}, y)\nu(D) \tag{3.1}$$

where the sum is over all Δ-floor diagrams such that $\delta(\mathcal{D}) = \delta$. Analogous to (1.38) we consider the generating function for these polynomials. That is

$$\mathcal{N}(S_\Delta, L_\Delta; y) := \sum_{\delta \geq 0} N_\delta(c, m, d; y)t^\delta. \tag{3.2}$$

The main goal of this chapter is to show that (3.2) is multiplicative in the parameters c, m, d defining the lattice polygon Δ. To be precise, we prove the following theorem.

Theorem 3.1. *Let (S_Δ, L_Δ) be $(\mathbb{P}^2, dH)$ or $(\mathbb{P}(1,1,m), dH)$ or $(\Sigma_m, cF + dH)$ then:*

(1) there are power series $S_0, \dots, S_6 \in \mathbb{Q}[y^{\pm 1}][\![t]\!]$, such that

$$\mathcal{N}((\Sigma_m, cF + dH); y) = S_0 S_1^c S_2^d S_3^{cd} S_4^m S_5^{md} S_6^{md^2};$$

(2) there are power series $P_{m,0}, P_{m,1}, P_{m,2} \in \mathbb{Q}[y^{\pm 1}][\![t]\!]$ such that for all $m \geq 1$

$$\mathcal{N}((\mathbb{P}(1,1,m), dH); y) = P_{m,0} P_{m,1}^d P_{m,2}^{d^2}.$$

In particular $\mathcal{N}(d, y) := \mathcal{N}((\mathbb{P}^2, dH); y) = P_{1,0} P_{1,1}^d P_{1,2}^{d^2}$.

Idea of Proof: We consider the formal logarithm of the generating function in (3.2) i.e. let

$$\mathcal{Q}(S_\Delta, L_\Delta; y) := \log \mathcal{N}(S_\Delta, L_\Delta; y) = \sum_{\delta \geq 1} Q_\delta(S_\Delta, L_\Delta; y)t^\delta.$$

Theorem 3.1 is equivalent to saying that $Q_\delta((\Sigma_m, cF+dH);y)$ and respectively $Q_\delta((\mathbb{P}(1,1,m),dH);y)$ is a linear combination of $1,c,d,cd,m,md,md^2$ and $1,d,d^2$. In §3.1 we begin by reviewing the *long edge graphs*, purely combinatorial graphs that will be central to proving this theorem. We also remark that Theorem 3.1, with a little bit care, can be generalized to toric surfaces that are defined by h-transverse lattice polygons that are not necessarily of the type $\Delta_{(c,m,d)}$.

In §3.2, we couple Theorem 3.1 with computer calculations to provide more evidence for the the conjectural generating function of the refined invariants [**GS14**, Conj. 62]. As already mentioned in the first and second chapter, the refined node polynomials specialize at $y=-1$ to the Welschinger invariants. We specialize the conjectural generating functions of the refined invariants to the Welschinger numbers. To be more precise:

Conjecture 3.2. *There exists universal power series $\overline{B}_1(q), \overline{B}_2(q) \in \mathbb{Q}[\![q]\!]$ such that*

$$\sum_{\delta\geq 0} W^{(S,L),\delta}\left(\overline{G}_2(q)\right)^\delta = \frac{(\overline{G}_2(q)/q)^{\chi(L)}\overline{B}_1(q)^{K_S^2}\overline{B}_2(q)^{LK_S}}{(\eta(q)^{16}\eta(q^2)^4 D\overline{G}_2(q)/q^2)^{\chi(\mathcal{O}_S)/2}}. \tag{3.3}$$

whenever L is a δ-very ample line bundle on S.

Conjecture 3.2 above has already stated in Chapter 1. Coupling Theorem 3.1 with computer calculations we provide evidence for the conjecture above for the case (S,L) is $(\mathbb{P}^2,dH)$, $(\mathbb{P}^1\times\mathbb{P}^1,dH)$ and $(\Sigma_m, cF+dH)$.

In §3.3 we extend the conjectures of of Göttsche and Shende [**GS14**] to singular toric surfaces. The conjectural generating function [**GS14**, Conj. 62] applies to smooth complex projective surfaces. On the other hand, the general version of Theorem 3.1 above applies to all projective toric surfaces defined by h-transverse lattice polygon. These toric surfaces are not necessarily smooth. Motivated by the paper [**LO14**], we extend the conjecture to singular toric surfaces.

CONJECTURE 3.3. *For every analytic type of singularities c there are formal power series $F_c \in \mathbb{Q}[y^{\pm 1}][[q]]$ such that the following hold. Let (S, L) be a pair of a projective toric surface and a toric line bundle on S. If L is δ-very ample on S, then*

$$\sum_{\delta \geq 0} N_\delta(S, L; y)\widetilde{DG_2}(y,q)^\delta = \frac{(\widetilde{DG_2}(y,q)/q)^{\chi(L)} B_1(y,q)^{K_S^2} B_2(y,q)^{LK_S}}{(\widetilde{\Delta}(y,q) D\widetilde{DG_2}(y,q)/q^2)^{\chi(\mathcal{O}_S)/2}} \prod_c F_c(y,q)^{n_c}. \tag{3.4}$$

Here c runs through the analytic types of singularities of S, and n_c is the number of singularities of S of type c.

We give a slightly more precise version of the above conjecture for $\mathbb{P}(1, 1, m)$ and its minimal resolution Σ_m and prove a special cases of it. Liu and Osserman [**LO14**] studied the non-refined Severi degrees for toric surfaces with only rational double points defined by h-transverse lattice polygons. We give conjectural generalization to their results to refined Severi degrees. Finally in §3.4 we consider a different kind of generalization to the conjectures of Göttsche and Shende. We consider refined versions of the problem of counting curves with prescribed multiple points.

3.1. Long Edge Graphs and the Multiplicativity Theorems

Brugallé and Mikhalkin [**BM07, BM09**] introduced the *marked labeled floor diagrams* and gave an enumerative formula for the unrefined Severi degrees in terms of the floor diagrams. Deleting edges of length 1 and weight 1 from a marking of a floor diagram then the resulting diagram is a long edge graph. The term long edge graph was first used in [**BCK14**], where the unrefined Severi degrees of a large class of toric surfaces is studied. We review long edge graphs from [**BCK14, Liu16, LO14**], working in the context of refined invariants and following the presentation in [**Liu16, LO14**].

DEFINITION 3.4. A *long edge graph* $G = (V, E)$ is a graph on a linearly ordered vertex set V and whose set of edges is endowed with a weight function $w : E \to \mathbb{Z}_{>0}$ and satisfying the following.

(1) The vertex set $V \subset \mathbb{Z}_{\geq 0}$ and the edge set E is finite.
(2) G is a multi-graph i.e. it can have have multiple edges connecting a pair of vertices. However, G may not have loops.

(3) G has no short edges. These are edges of weight 1 connecting i and $i+1$.

An edge connecting i and j with $i<j$ will be denoted $(i \to j)$ and the *length* of an edge $e=(i\to j)$ is $\ell(e) := j-i$. The vertices $0,1,2,\dots$ of long edge graphs are drawn from left to right and each edge is labeled with its weight, unless it is of weight 1 in which case the labeling is suppressed. Since the edge set E is finite, we often omit the vertices that are not incident to any edge.

Definition 3.5. Let $G=(V,E)$ be a long edge graph and let $w: E \to \mathbb{Z}_{>0}$ be a weight function on the set of edges of the graph. The refined multiplicity of G is defined to be

$$\mu(G,y) := \prod_{e\in E} \left([w(e)]_y\right)^2,$$

where for an integer n, $[n]_y$ is the Laurent polynomial introduced in Remark 1.19. The Severi multiplicity and the Welschinger multiplicity of G are given respectively by $\mu(G,1)$ and $\mu(G,-1)$. The cogenus of G is defined to be

$$\delta(G) := \sum_{e\in E} \big(\ell(e)w(e)-1\big).$$

We denote by $\operatorname{minv}(G)$ and $\operatorname{maxv}(G)$ the smallest and respectively the largest vertex i of G that is incident to at least one edge. The length of G is defined to be $\ell(G) := \operatorname{maxv}(G) - \operatorname{minv}(G)$. For an integer $k \geq 0$, we denote by $G_{(k)}$ the graph obtained by shifting edges of G to the right by k.

Example 3.6. Figure 3.1 shows some examples of long-edge graphs. H is obtained from G by shifting it to the right by 3 i.e. $H = G_{(3)}$. Thus G and H have the same length, same refined multiplicity and cogenus. These are given respectively by $\ell(G) = \ell(H) = 2$, $\delta(G) = \delta(H) = 2$ and

$$\mu(G,y) = \mu(H,y) = y + 2 + y^{-1}.$$

Further, we have $\operatorname{minv}(K) = 3, \operatorname{maxv}(K) = 6, \delta(K) = 3$ and $\mu(K,y) = y^{-2} + 4y^{-1} + 6 + 4y + y^2$.

Definition 3.7. Let $G=(V,E)$ be a long-edge graph and $e: E \to \mathbb{Z}_{>0}$ be a weight function on its edges. For any vertex j of G let

$$\lambda_j = \lambda_j(G) := \sum_{e} w(e),$$

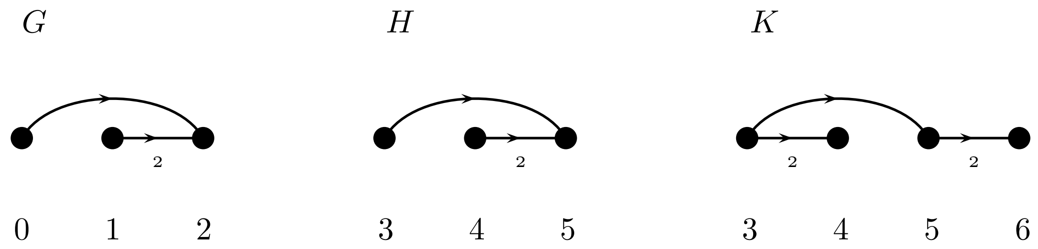


FIGURE 3.1. Examples of long-edge graphs.

where the sum is over all edges $i \xrightarrow{e} k$ such that $i < j \leq k$.

DEFINITION 3.8. Let $\beta = (\beta_1, \ldots, \beta_{M+1}) \in (\mathbb{Z}_{\geq 0})^{M+1}$ be a sequence of integers. G is called *β-allowable* if $\mathrm{maxv}(G) \leq M+1$ and $\beta_j \geq \lambda_j(G)$ for all $j = 1, \ldots, M+1$. G is called *strictly β-allowable* if it is β-allowable and furthermore all edges incident to 0 or $M+1$ have weight 1. Also write $\overline{\lambda}_j(G) := \lambda_j(G) - \#\{\text{edges } (j-1 \to j)\}$. G is called *β-semiallowable* if $\mathrm{maxv}(G) \leq M+1$ and $\beta_j \geq \overline{\lambda}_j(G)$ for all j.

DEFINITION 3.9. Let G be a long edge graph. Let $\epsilon_0(G) := 1$, if all edges adjacent to $\mathrm{minv}(G)$ have weight 1, and $\epsilon_0(G) := 0$ otherwise. Similarly let $\epsilon_1(G) := 1$, if all edges adjacent to $\mathrm{maxv}(G)$ have weight 1, and $\epsilon_1(G) := 0$ otherwise.

DEFINITION 3.10. A long edge graph Γ is said to be a *template* if for any vertex $1 \leq i \leq \ell(\Gamma) - 1$ there exists at least one edge $(j \to k)$ with $j < i < k$. A long edge graph G is said to be a *shifted template* if $G = \Gamma_{(k)}$ for some template $k \in \mathbb{Z}_{\geq 0}$.

The following lemma will be useful in the proof of the main theorem. It has already been used (without proof) in [**Liu16, LO14**].

LEMMA 3.11. *Let Γ be a template and let $\delta(\Gamma)$ and $l(\Gamma)$ be its cogenus and length respectively then $\delta(\Gamma) \geq l(\Gamma) - \epsilon_1(\Gamma)$.*

PROOF. The proof is by induction on $l(\Gamma)$. Assume that $l(\Gamma) = 0$, then the result holds trivially. Let Γ_0 be a template consisting of a single edge e. By definition, $\delta(\Gamma_0) = \rho(e)l(e) - 1$. We have the following cases

$$\begin{cases} \delta(\Gamma_0) = l(e) - 1 \text{ if } \rho(e) = 1 \\ \delta(\Gamma_0) = \rho(e)l(e) - 1 \geq l(e) \text{ otherwise.} \end{cases} \tag{3.5}$$

In both cases, it follows therefore that $\delta(\Gamma_0) \geq l(\Gamma_0) - \epsilon_1(\Gamma_0)$. Let Γ be any template and let $e = j \xrightarrow{e} \mathrm{maxv}(\Gamma)$ be the longest edge adjacent to $\mathrm{maxv}(\Gamma)$. If there is more than one such edge then choose one arbitrarily. Let Γ' be the graph obtained from Γ by deleting all edges $i \to k$ with $j \leq i$ except e. By construction we have $l(\Gamma) = l(\Gamma')$, $\epsilon_1(\Gamma') \geq \epsilon_1(\Gamma)$ and moreover

$$\begin{cases} \delta(\Gamma) > \delta(\Gamma') \text{ if } \epsilon_1(\Gamma') > \epsilon_1(\Gamma) \\ \delta(\Gamma) \geq \delta(\Gamma') \text{ otherwise.} \end{cases} \tag{3.6}$$

Let Γ'' be the graph obtained from Γ' by deleting e. If Γ'' is empty then by (3.6) and (3.5) above the result follows. So we can assume that Γ'' in not empty. Then Γ'' is a template such that $l(\Gamma'') < l(\Gamma)$ and using properties of templates and (3.5) above one finds that

$$l(e) + l(\Gamma'') > l(\Gamma) \text{ and } \delta(\Gamma') = \delta(\Gamma'') + \rho(e)l(e) - 1 \geq \delta(\Gamma'') + l(e) - \epsilon_1(\Gamma'). \tag{3.7}$$

This implies that $\delta(\Gamma') \geq l(\Gamma) + 1 - \epsilon_1(\Gamma')$. Since $l(\Gamma'') < l(\Gamma)$ by induction we have that $\delta(\Gamma'') \geq l(\Gamma'') - \epsilon_1(\Gamma'')$. Finally using (3.6) and (3.7) above we have

$$\begin{cases} \delta(\Gamma) > l(\Gamma) + 1 - \epsilon_1(\Gamma') \text{ if } \epsilon_1(\Gamma') > \epsilon_1(\Gamma) \\ \delta(\Gamma) \geq l(\Gamma) + 1 - \epsilon_1(\Gamma') \text{ if } \epsilon_1(\Gamma') = \epsilon_1(\Gamma). \end{cases} \tag{3.8}$$

In the first case, $\epsilon_1(\Gamma') > \epsilon_1(\Gamma)$ implies that $\epsilon_1(\Gamma) = 0$ and $\epsilon_1(\Gamma') = 1$ and therefore $\delta(\Gamma) > l(\Gamma)$. The second case is straight forward hence the result. ∎

Definition 3.12. Let $\beta = (\beta_1, \ldots, \beta_{M+1}) \in (\mathbb{Z}_{\geq 0})^{M+1}$ and G be a β-allowable long-edge graph. Create a new graph $\mathrm{ext}_\beta(G)$ by adding $\beta_j - \lambda_j(G)$ edges of weight 1 connecting $j-1$ and j for all $j = 1, \ldots, M+1$. A *β-extended ordering* of G is a total ordering on the union of the vertices and edges of $\mathrm{ext}_\beta(G)$, such that

(1) it extends the natural ordering of the vertices $0, 1, 2, \ldots$,
(2) if an edge e connects vertices i and j, then e is between i and j.

Two extended orderings o, o' of G are considered equivalent if there is an automorphism σ of the edges, permuting only edges connecting the same vertices and of the same weight such that $\sigma(o) = o'$.

DEFINITION 3.13. Let $\beta = (\beta_1, \ldots, \beta_{M+1}) \in (\mathbb{Z}_{\geq 0})^{M+1}$ and G be a β-allowable long-edge graph. Denote by $P_\beta(G)$ the number of β-extended orderings of G up to equivalence. Here $P_\beta(G)$ is defined to be 0, if G is not β-allowable. Furthermore let

$$P^s_\beta(G) := \begin{cases} P_\beta(G) & G \text{ strictly } \beta\text{-allowable}, \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE 3.14. Let $\beta = (\beta_1, \beta_2, \beta_3) = (3, 4, 5)$. For this β, the graph G in Figure 3.1 is strictly β-allowable (note that $\lambda_1(G) = 1, \lambda_2(G) = 3$ and $\lambda_j(G) = 0$ for all other $j \neq 1, 2$). From Figure 3.2 it is easy to see that $P^s_\beta(G) = 12$.

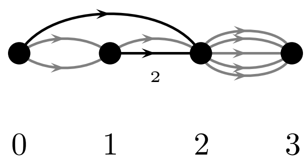


FIGURE 3.2. The extended graph $\mathrm{ext}_\beta(G)$.

DEFINITION 3.15. Given any $\delta \geq 0$ and any $\beta \in (\mathbb{Z}_{\geq 0})^{M+1}$ define

$$N^\delta_\beta(y) := \sum_G \mu(G, y) P^s_\beta(G), \quad n^\delta_\beta := \sum_G \mu(G, 1) P^s_\beta(G), \quad W^\delta_\beta := \sum_G \mu(G, -1) P^s_\beta(G)$$

where the sum is over all long edge graphs such that $\delta(G) = \delta$.

Let $\beta = (\beta_1, \ldots, \beta_{d+1})$. We introduce the terminology of *β-graphs* for the sole purpose of proving Theorem 3.18 below. A β-graph G is defined precisely like a long-edge graph, except that we allow short edges and require that $\beta_j = \lambda_j(G)$ for all $j = 1, \ldots, d+1$ where $\lambda_j(G)$ is defined similarly as in the case of long-edge graphs. Furthermore, we define the multiplicity and the cogenus of a β-graph just as we would for a long-edge graph.

LEMMA 3.16. *Let $\beta \in \mathbb{Z}^{d+1}_{\geq 0}$. There exists a cogenus preserving bijection between strictly β-allowable long-edge graphs and β-graphs whose edges incident to* 0 *or* $d+1$ *have weight 1.*

PROOF. Let G be a strictly β-allowable long-edge graph. The map $G \mapsto \mathrm{ext}_\beta(G)$ associate to G a β-graph. It is clear from definition that $\mathrm{ext}_\beta(G)$ is a β-graph whose edges adjacent to 0 or $d+1$ have weight 1. On the other hand, removing short edges from a β-graph H yields a long-edge graph which is strictly β-allowable if and only if all edges of H incident to 0 or $d+1$ have weight 1. Furthermore, the fact that short edges contribute 0 to the cogenus means that $\delta(G) = \delta(\mathrm{ext}_\beta(G))$. Conversely let H' be the graph obtained after removing short edges from a β-graph H then its also clear that H' is a long-edge graph satisfying $\delta(H') = \delta(H)$. $\square$

Let $c, m, d \in \mathbb{Z}_{\geq 0}$. For our purposes, we are interested in sequences in $(\mathbb{Z}_{\geq 0})^{d+1}$ of the following type: put $s(c, m, d) := (e_0, \ldots, e_d)$ where $e_i = c + mi$. Let $s_1, \ldots, s_d$ be a sequence of non-negative integers such that $s_1 + \cdots + s_d = c$. In the following we will consider β-graphs H such that:

($\star$) for each $j : 1 \leq j \leq d$ then H has precisely s_j edges of weight 1 of type $0 \xrightarrow{e} j$ and additionally all edges of type $j \xrightarrow{e} d+1$ are of weight 1 and the number of such edges is less or equal to $m + s_j$.

Lemma 3.17. *Let $\Delta = \Delta_{(c,m,d)}$ for $c, m, d \in \mathbb{Z}_{\geq 0}$ and let $\beta = \beta(\Delta) := s(c, m, d)$. There exists a cogenus preserving bijection between β-graphs satisfying ($\star$) above and the markings of Δ-floor diagrams.*

Proof. Let $\mathcal{D}$ be a Δ-floor diagram and consider the graph obtained after applying Step 1 and Step 2 of Definition 2.35. Then identifying all vertices created in Step 1 to a vertex 0 and all the vertices created in Step 2 to a vertex $d+1$ then it is easy to see that the resulting graph $G(\mathcal{D})$ is a β-graph satisfying ($\star$) above. On the other hand let H be a β-graph and denote by $\widetilde{H}$ the graph obtained from H by deleting vertices 0 and $d+1$ and all the edges that are incident to these vertices. Then $\widetilde{H}$ is a graph on the vertex set $\{1, \ldots, d\}$ satisfying the following. For $j : 1 \leq j \leq d$ let s_j be the number of edges of type $0 \xrightarrow{e} j$ then

$$\mathrm{div}(j) = \beta_{j+1} - \beta_j = c + m(j+1) - c + mj = m \leq m + s_j.$$

In other words, $\widetilde{H}$ is a Δ-floor diagram. Next, we show that this correspondence preserves cogenus. Let: E be the edge set of $\mathcal{D}$; $\widetilde{E}$ be the edge set of a marking $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ and $G(E)$ be the edge set of $G(\mathcal{D})$. First note that for $\Delta = \Delta_{(c,m,d)}$ we have that

$$\#(\mathrm{int}(\Delta) \cap \mathbb{Z}^2) = cd - c - dm + m(d(d+1))/2 - d + 1. \tag{3.9}$$

Using the fact that for a floor diagram $\mathcal{D}$ we have $\sum_{j=1}^{d}\operatorname{div}(j)=0$ we see that the cardinality of $\widetilde{E}$ is $2\#E+2c+dm$ whereas the number of vertices of $\widetilde{\mathcal{D}}$ is $d+\#E+2c+dm$. Thus by definition $\delta(\widetilde{\mathcal{D}})=\#(\operatorname{int}(\Delta)\cap\mathbb{Z}^2)-\#E+d-1$. Note in particular that $\delta(\mathcal{D})=\delta(\widetilde{\mathcal{D}})$ i.e. the map $\mathcal{D}\mapsto\widetilde{\mathcal{D}}$ preserves the cogenus. On the other hand the number of edges in $G(E)$ is $\#E+2c+dm$ and thus

$$\delta(G(\mathcal{D}))=\sum_{e\in G(E)}w(e)l(e)-\#G(E)=\sum_{j=0}^{d}(c+mj)-\#E-2c-dm$$

Simplifying the right hand side of the above equation we get

$$cd-c-dm+m(d(d+1))/2-\#E=\#(\operatorname{int}(\Delta)\cap\mathbb{Z}^2)+d-1-\#E.$$

This shows that $\delta(G(\mathcal{D}))=\delta(\widetilde{\mathcal{D}})$. □

The relation of $N_\beta^\delta(y)$ to the refined Severi degrees and the tropical Welschinger invariants is given in the following theorem.

THEOREM 3.18. *Let S be $\mathbb{P}^2,\mathbb{P}(1,1,m)$ or Σ_m and L be a line bundle on S. On $\mathbb{P}^2,\mathbb{P}(1,1,H)$ let H be the hyperplane bundle and on Σ_m let $H:=E+mF$ where F is the class of the ruling and E is the class of a section with $E^2=-m$. Then*

(1) for the refined Severi degrees

$$N^{d,\delta}(y)=N^\delta_{s(0,1,d)}(y),\quad N^{(\mathbb{P}(1,1,m),dH),\delta}(y)=N^\delta_{s(0,m,d)}(y)\ \textit{and}$$
$$N^{(\Sigma_m,cF+dH),\delta}(y)=N^\delta_{s(c,m,d)}(y),$$

(2) for the Severi degrees

$$n^{d,\delta}=n^\delta_{s(0,1,d)},\quad n^{(\mathbb{P}(1,1,m),dH),\delta}=n^\delta_{s(0,m,d)}\ \textit{and}\ n^{(\Sigma_m,cF+dH),\delta}=n^\delta_{s(c,m,d)},$$

(3) and for the Welschinger invariants

$$W^{d,\delta}=W^\delta_{s(0,1,d)},\quad W^{(\mathbb{P}(1,1,m),dH),\delta}=W^\delta_{s(0,m,d)}\ \textit{and}\ W^{(\Sigma_m,cF+dH),\delta}=W^\delta_{s(c,m,d)}.$$

PROOF. For any toric surface S and toric line bundle L on S we know that $n^{(S,L),\delta}=N^{(S,L),\delta}(1)$ and $W^{(S,L),\delta}=N^{(S,L),\delta}(-1)$. On the other hand by Definition 3.5 and Definition 3.15 we have $n_\beta^\delta=N_\beta^\delta(1)$ and $W_\beta^\delta=N_\beta^\delta(-1)$. Therefore to prove the theorem it is enough to consider case (1) only. Further, it is enough to consider the case $S = \Sigma_m$ since by Definition 2.36 and Theorem 2.37 we have $N^{(\mathbb{P}(1,1,m),dH),\delta}(y) = N^{(\Sigma_m,dH),\delta}(y)$. The idea is to show that

$$N^\delta_\beta(y) = \sum_{\mathcal{D}} \mu(\mathcal{D}, y)\nu(\mathcal{D}) \tag{3.10}$$

where the sum is over all Δ-floor diagrams such that $\delta(\mathcal{D}) = \delta$. By Lemma 3.16 and Lemma 3.17, there exists a bijection between (markings of) Δ-floor diagrams and strictly allowable long-edge graphs which respects cogenus. Furthermore, let $G(D)$ be as in Lemma 3.17 above then the number of β-extended orderings of $G(D)$ is clearly equal to the number of inequivalent markings of $\mathcal{D}$. This is precisely (3.10) above. □

Consider the generating function for $N^{\Delta,\delta}(y)$ which for $\Delta = \Delta_{c,m,d}$ and by Theorem 3.18 above can be expressed as

$$\mathcal{N}((S_\Delta, L_\Delta), y) := 1 + \sum_{\delta\geq 1} N^{(S_\Delta,L_\Delta),\delta}(y)t^\delta = 1 + \sum_{\delta\geq 1} N^\delta_\beta(y)t^\delta. \tag{3.11}$$

Consider the formal logarithm of the above generating function, that is

$$\log(1 + \sum_{\delta\geq 1} N^\delta_\beta(y)t^\delta) = \sum_{\delta\geq 1} Q^\delta_\beta(y)t^\delta.$$

To prove Theorem 3.1 is equivalent to proving that $Q^\delta_\beta(y)$ is linear in the parameters defining Δ as already explained in the introductory section. This leads to a consideration of the logarithmic version of $P_\beta(G)$ and $P^s_\beta(G)$. Again we follow the notations and definitions in [**Liu16, LO14**].

Definition 3.19. A *partition* of a long edge graph $G = (V, E, w)$ is a tuple $(G_1, \dots, G_n)$ of nonempty long edge graphs such that the disjoint union of the (weighted) edge sets of $G_1, \dots, G_n$ is the (weighted) edge set of G.

For any long edge graph define

$$\Phi_\beta(G) := \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \sum_{G_1,\dots,G_n} \prod_{j=1}^n P_\beta(G_j),$$

$$\Phi^s_\beta(G) := \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \sum_{G_1,\dots,G_n} \prod_{j=1}^n P^s_\beta(G_j),$$

where both summations are over the partitions of G. Using the same arguments used in the unrefined setting [**LO14**, §3] we have that

$$Q^\delta_\beta(y) = \sum_G \mu(G, y)\Phi^s_\beta(G), \tag{3.12}$$

where the sum is taken again over all long-edge graphs of cogenus δ. By [**Liu16**, Lem. 2.15] we have $\Phi^s_\beta(G) = 0$, if G is not a shifted template. On the other hand [**Liu16**, Cor. 3.5] says that for a template Γ and for $\beta = (\beta_1, \ldots, \beta_{M+1}) \in (\mathbb{Z}_{\geq 0})^{M+1}$ we have

$$\Phi^s_\beta(\Gamma_{(k)}) = \begin{cases} \Phi_\beta(\Gamma_{(k)}) & 1 - \epsilon_0(\Gamma) \leq k \leq M + \epsilon_1(\Gamma) - \ell(\Gamma) \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

Using (3.13) together with (3.12), we obtain the following refined version of [**LO14**, Cor. 3.6].

Corollary 3.20. *Let $\beta = (\beta_1, \ldots, \beta_{M+1}) \in \mathbb{Z}^{M+1}_{\geq 0}$. Then*

$$Q^\delta_\beta(y) = \sum_\Gamma \mu(\Gamma, y) \sum_{k=1-\epsilon_0(\Gamma)}^{M-\ell(\Gamma)+\epsilon_1(\Gamma)} \Phi_\beta(\Gamma_{(k)}),$$

where the first sum runs over all templates Γ of cogenus δ.

The following theorem will be an ingredient to the proof of Theorem 3.22 below. It is included here for completeness.

Theorem 3.21. *([**LO14**, Thm. 3.8]) Let G be a long edge graph. There exists a linear multivariate function $\Phi(G, \beta)$ in β, such that for any β such that G is β-semiallowable, we have $\Phi_\beta(G) = \Phi(G, \beta)$. Furthermore writing $\beta = (\beta_0, \ldots, \beta_M) \in \mathbb{Z}^{M+1}_{\geq 0}$, the linear function $\Phi(G, \beta)$ is a linear combination of the β_i with $\mathrm{minv}(G) \leq i \leq \mathrm{maxv}(G)$.*

Now we prove the equivalent statement to Theorem 3.1. This will show that the generating functions for the refined Severi degrees on weighted projective spaces and rational ruled surfaces are multiplicative. Similar strategy has been used in [**BCK14, Liu16**]. In particular, in [**Liu16**, Thm. 1.4] the multiplicativity of the generating function for the Severi degrees on $\mathbb{P}^2$ is proven by showing that Q^δ_β is quadratic in d(to be precise linear in $1, d, d^2$).

THEOREM 3.22. *Let* $\Delta = \Delta_{(c,m,d)}$ *such that* (S_Δ, L_Δ) *be* $(\mathbb{P}^2, dH)$ *or* $(\mathbb{P}(1,1,m), dH)$ *or* $(\Sigma_m, cF + dH)$. *Then* $Q^{(S_\Delta, L_\Delta),\delta}(y)$ *can be expressed as a linear combination of* $1, c, d, cd, m, md, md^2$. *To be precise*

(1) *if* $c \geq \delta$ *and* $d \geq \delta$, *then* $Q^{(\Sigma_m, cF+dH),\delta}(y)$ *is a* $\mathbb{Q}[y^{\pm 1}]$*-linear combination of* 1, c, d, cd, m, md, md^2;
(2) *in particular if* $c \geq \delta$, $d \geq \delta$, *then* $Q^{(\mathbb{P}^1\times\mathbb{P}^1, cF+dH),\delta}(y)$ *is a* $\mathbb{Q}[y^{\pm 1}]$*-linear combination of* 1, $c+d$, cd.
(3) *Fix* $m \geq 1$, $c \geq 0$. *If* $d \geq \delta$ *then* $Q^{(\Sigma_m, dH+cF),\delta}(y)$ *is a polynomial of degree* 2 *in* d.
(4) *Fix* $m \geq 1$. *If* $d \geq \delta$, *then* $Q^{(\mathbb{P}(1,1,m),dH),\delta}(y)$ *is a polynomial of degree* 2 *in* d. *In particular for* $d \geq \delta$, $Q^{d,\delta}(y)$ *is a polynomial of degree* 2 *in* d.
(5) *If* $d, m \geq \delta$, *then* $Q^{(\mathbb{P}(1,1,m),dH),\delta}(y)$ *is a* $\mathbb{Q}[y^{\pm 1}]$*-linear combination of* 1, m, d, dm, d^2m.

PROOF. (1) By Theorem 3.18 above we that have $Q^{(\Sigma_m, cF+dH),\delta}(y) = Q^\delta_{s(c,m,d)}(y)$. Consequently, Corollary 3.20 implies that

$$Q^{(\Sigma_m, cF+dH),\delta}(y) = \sum_\Gamma \mu(\Gamma, y) \sum_{k=1-\epsilon_0(\Gamma)}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)} \Phi_{s(c,m,d)}(\Gamma_{(k)}), \tag{3.14}$$

where the first sum is over all templates Γ such that $\delta(\Gamma) = \delta$.

Let Γ now be a template of cogenus δ, and let k be an integer in the interval $[1 - \epsilon_0(\Gamma), d - \ell(\Gamma) + \epsilon_1(\Gamma)]$. Then by definition we get

$$\Phi_{s(c,m,d)}(\Gamma_{(k)}) = \Phi_{s(c+km,m,\ell(\Gamma)-1)}(\Gamma).$$

On the other hand using [**LO14**, Lem. 4.2] we have $\overline{\lambda}_i(\Gamma) \leq \delta$ for all i. By our assumption we have $c \geq \delta \geq \overline{\lambda}_i(\Gamma)$, which imply that

$$c + m(k+i) \geq \overline{\lambda}_i(\Gamma)$$

for all i. In other words, Γ is $s(c+km, m, \ell(\Gamma)-1)$-semiallowable. Therefore, by Theorem 3.21, it follows that $\Phi_{s(c+km,m,\ell(\Gamma)-1)}(\Gamma)$ is a linear function in $c + lm$, for l in the interval $k \leq l \leq k + \ell(\Gamma) - 1$, thus it is linear function in c, m and km of the form $\alpha + \beta(c + km) + \gamma m$, with $\alpha, \beta, \gamma \in \mathbb{Q}$.

Define $M_1 := d - \ell(\Gamma) + \epsilon_1(\Gamma) + \epsilon_0(\Gamma)$, $M_2 := d - \ell(\Gamma) + \epsilon_1(\Gamma) - \epsilon_0(\Gamma) + 1$. By Lemma 3.11 we have $\ell(\Gamma) - \epsilon_1(\Gamma) \leq \delta$, so, by our assumption $d \geq \delta$, we have

$M_1 \geq 0$. Recall that for integers a, b such that $b \geq 0$ and $b \geq a-1$ we have the trivial identity

$$\sum_{k=a}^{b} k = \frac{(a+b)(b-a+1)}{2}. \tag{3.15}$$

Thus we get

$$\begin{aligned}\sum_{k=1-\epsilon_0(\Gamma)}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)} \Phi_{s(c,m,d)}(\Gamma_{(k)}) &= \sum_{k=1-\epsilon_0(\Gamma)}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)} \big(\alpha + \beta(c+km) + \gamma m\big) \\ &= M_1(\alpha + \beta c + \gamma m) + \frac{M_1 M_2}{2}\beta m,\end{aligned}$$

which is a $\mathbb{Q}$-linear combination of $1, c, d, cd, m, md, md^2$. Consequently (3.14) implies that $Q^{(\Sigma_m, cF+dH),\delta}(y)$ is a $\mathbb{Q}[y^{\pm 1}]$ linear combination of $1, c, d, cd, m, md, md^2$.

(2) We use the fact that $\mathbb{P}^1 \times \mathbb{P}^1$ is defined by the polygon $\Delta = \Delta_{(c,m,d)}$ with $m = 0$. By case (1) above it follows therefore that $Q^{(\mathbb{P}^1\times\mathbb{P}^1, cF+dH),\delta}(y)$ is a linear combination of 1, c, d, cd. It is clearly symmetric under exchange of c and d, and thus a linear combination of 1, $c+d$, cd.

(3) We proceed with a similar strategy as in the proof of case (1). Here, $c \geq 0$ is fixed and not necessarily larger that δ and therefore, we need a different strategy to prove semiallowability of the template. Again by Corollary 3.20 and Theorem 3.18 we have,

$$Q^{(\Sigma_m, cF+dH),\delta}(y) = Q^{\delta}_{s(c,m,d)}(y) = \sum_{\Gamma} M(\Gamma) \sum_{k=1-\epsilon_o(\Gamma)}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)} \Phi_{s(c,m,d)}(\Gamma_{(k)}), \tag{3.16}$$

with Γ again running through all templates of cogenus δ.

Let Γ be a template of cogenus δ, and let k be an integer lying in the interval $[1-\epsilon_0(\Gamma), d-\ell(\Gamma)+\epsilon_1(\Gamma)]$. Then again using the definition of a shifted template we get $\Phi_{s(c,m,d)}(\Gamma_{(k)}) = \Phi_{s(c+km,m,\ell(\Gamma)-1)}(\Gamma)$. For a rational number a we denote by $\lceil a \rceil$ the smallest integer bigger or equal to a. We put

$$k_{min} := \max\left(1, \max\left(\left\lceil \frac{\overline{\lambda}_i(\Gamma)}{m} \right\rceil - i + 1 \,\middle|\, i = 1, \dots, \ell(\Gamma)\right)\right).$$

This implies that for any $k \geq k_{\min}$ then $c + m(k+i-1) \geq \overline{\lambda}_i(\Gamma)$ for all i. Therefore Γ is $s(c+km, m, \ell(\Gamma)-1)$-semiallowable and thus for $k \geq k_{min}$, we have that $\Phi_{s(c+km,m,\ell(\Gamma)-1)}(\Gamma)$ is a linear function in the lm, $k \leq l \leq k+\ell(\Gamma)-1$, thus it is a linear function $\alpha + \beta km + \gamma m$, with $\alpha, \beta, \gamma \in \mathbb{Q}$.

By [**LO14**, Lem. 4.2], we have $\overline{\lambda}_i(\Gamma) \leq \delta - \ell(\Gamma) + i + \epsilon_1(\Gamma)$. As $\overline{\lambda}_i(\Gamma) \geq 0$, this implies

$$\left\lceil \frac{\overline{\lambda}_i(\Gamma)}{m} \right\rceil - i + 1 \leq \delta + \epsilon_1(\Gamma) - \ell(\Gamma) + 1$$

for all i. By the inequality $\ell(\Gamma) - \epsilon_1(\Gamma) \leq \delta$, already used in part (1), this implies $k_{min} \leq \delta + \epsilon_1(\Gamma) - \ell(\Gamma) + 1$. By our assumption $d \geq \delta$, we have $d - \ell(\Gamma) + \epsilon_1(\Gamma) - k_{min} + 1 \geq 0$. Therefore using (3.15) then the sum

$$\sigma(\Gamma, k_{min}) := \sum_{k=k_{min}}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)} \Phi_{s(c,m,d)}(\Gamma_{(k)})$$

is a $\mathbb{Q}$-linear combination of 1, d, m, md, md^2. If we fix m, it is a linear combination of 1, d, d^2. But

$$\sum_{k=1-\epsilon_0(\Gamma)}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)} \Phi_{s(c+km,m,l(\Gamma)-1)}(\Gamma) = \sigma(\Gamma, k_{min}) + \sum_{k=1-\epsilon_0(\Gamma)}^{k_{min}-1} \Phi_{s(c+km,m,l(\Gamma)-1)}(\Gamma).$$

The second sum is for fixed m just a finite number, thus the claim follows.

(4) As $Q^{(\mathbb{P}(1,1,m),dH),\delta}(y) = Q^{(\Sigma_m,dH),\delta}(y)$, (4) is a special case of (3).

(5) By Corollary 3.20 and Theorem 3.18,

$$Q^{(\mathbb{P}(1,1,m),dH),\delta}(y) = Q^{\delta}_{s(0,m,d)}(y) = \sum_{\Gamma} M(\Gamma) \sum_{k=1}^{d-\ell(\Gamma)+\epsilon_1(\Gamma)} \Phi_{s(0,m,d)}(\Gamma_{(k)}), \tag{3.17}$$

with Γ running through all templates of cogenus δ. According to Corollary 3.20, the inner sum starts at $k = 1 - \epsilon_0(\Gamma)$. But Γ is a template and therefore not $s(0, m, d)$-semiallowable. Thus (in case $\epsilon_0(\Gamma) = 1$), the contribution for $k = 0$ vanishes.

We have $Q^{(\mathbb{P}(1,1,m),dH),\delta}(y) = Q^{\delta}_{s(0,m,d)}(y)$, which is computed by the case $c = 0$ of (3.17). $k = 1$, because a template Γ can never be $(0, m, d)$-semiallowable and thus (in case $\epsilon_0(\Gamma) = 1$), the contribution for $k = 0$ vanishes. If $m \geq \delta$, then $k_{min} = 1$ for all templates Γ of cogenus δ, thus

$$Q^{(\mathbb{P}(1,1,m),dH),\delta}(y) = Q^{\delta}_{s(0,m,d)}(y) = \sum_{\Gamma} M(\Gamma)\sigma(\Gamma, 1),$$

with Γ again running through the templates of cogenus δ. By (3) this is a $Q[y^{\pm 1}]$-linear combination of 1, d, m, md, md^2. $\square$

REMARK 3.23. Theorem 3.1 above does not in general proof the multiplicativity of the generating function for the refined node polynomials on S_Δ with $\Delta = \Delta_{(c,m,d)}$.

Recall that multiplicativity means that for the pair (S, L) of a smooth projective surface and a line bundle then

$$\mathcal{N}((S,L);y) := \sum_{\delta\geq 0} N^{(S,L),\delta}(y)t^\delta = A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{\chi(\mathcal{O}_S)}$$

for some $A_i \in \mathbb{Q}[y^{\pm 1}][\![t]\!]$. Defining $A_1 := P_{1,2}, A_2^{-3} := P_{1,1}$ and $A_3^9 A_4 = P_{1,0}$ then Theorem 3.1 shows that $\mathcal{N}(d, y)$ has a multiplicative structure. More needs to be done to show the same for $\mathcal{N}((\Sigma_m, cF + dH); y)$.

Recall that a generic line bundle on a ruled surface Σ_m is of the form $L = cF+dH$ where F is the class of the ruling and $H = E+mF$ and E is the class with $E^2 = -m$. In particular we have that $K_{\Sigma_m} = -2H + (m-2)F$. Thus on (Σ_m, L) we have

$$(L^2, LK_{\Sigma_m}, K_{\Sigma_m}^2, \chi(\mathcal{O}_S)) = (md^2 + 2cd, -md - 2c - 2d, 8, 1).$$

Theorem 3.1 above states that $\mathcal{N}((\Sigma_m, cF + dH); y) = S_0 S_1^c S_2^d S_3^{cd} S_4^m S_5^{md} S_6^{md^2}$. To show multiplicativity one would therefore need to show that

- $S_6 = S_3^2$ which by universality would further imply $A_1 = P_{1,2} = S_6 = S_3^2$,
- $A_2 = S_1^{-2} = S_2^{-2} S_5^{-1} = P_{1,1}^{-1/3}$,
- $A_3^8 A_4 = S_0 S_4^m$.

REMARK 3.24. Theorem 3.22 above can be generalized to a large class of toric surfaces defined by h-transverse lattice polygons. For a sequence $\boldsymbol{s} = (s_1, \ldots, s_M)$ the *reversal set* of s is defined (see [**LO14**, Def. 2.9]) to be

$$\mathrm{Rev}(s) = \{1 \leq i < j \leq M : s_i < s_j\}.$$

Let Δ be a general h-transverse lattice polygon defined by the quadruple $\{\boldsymbol{l}, \boldsymbol{r}, d^t, d^b\}$ where $\boldsymbol{l} = (l_1, \ldots, l_h)$ and $\boldsymbol{r} = (r_1, \ldots, r_h)$ are the left and respectively the right directions associated to Δ. See §2.4 of Chapter 2 for a brief discussion. Define the cogenus of the pair $(\boldsymbol{l}, \boldsymbol{r})$ to be

$$\delta(\boldsymbol{l}, \boldsymbol{r}) := \sum_{(i,j)\in \mathrm{Rev}(\boldsymbol{r})} (r_j - r_i) + \sum_{(i,j)\in \mathrm{Rev}(-\boldsymbol{l})} (l_i - l_j).$$

Ardila and Block [**AB13**, Prop. 3.3] showed that for the Severi degrees on general toric surfaces defined by h-transverse lattice polygon Δ associated to the quadruple

$\{\boldsymbol{l}, \boldsymbol{r}, d^t, d^b\}$ then

$$N^{\Delta,\delta} = \sum_{(\boldsymbol{l},\boldsymbol{r})} \sum_{G} \mu(G) P^s_{\beta(d^t,\boldsymbol{r}-\boldsymbol{l})}(G) \tag{3.18}$$

where the first sum is over all reorderings $\boldsymbol{l} = (l_1, \ldots, l_h)$ and $\boldsymbol{r} = (r_1, \ldots, r_h)$ of the left and the right directions of Δ satisfying $\delta(\boldsymbol{l}, \boldsymbol{r}) \leq \delta$, where the second sum is over all long-edge graphs G of cogenus $\delta - \delta(\boldsymbol{l}, \boldsymbol{r})$ and where for a sequence $t = (t_0, t_1, \ldots, t_M)$ then $\beta(t) = (t_0, t_0 + t_1, \ldots, t_0 + \cdots + t_M)$. Counting every curve with its refined multiplicity then following the steps of Theorem 3.18 carefully above yields the refined version of (3.18) above

$$N^{\Delta,\delta}(y) = \sum_{(\boldsymbol{l},\boldsymbol{r})} N^{\delta-\delta(\boldsymbol{l},\boldsymbol{r})}_{\beta(d^t,\boldsymbol{r}-\boldsymbol{l})}(y). \tag{3.19}$$

Specializing (3.19) above at $y = 1$ gives (3.18) while specializing it at $y = -1$ gives

$$W^{\Delta,\delta} = \sum_{(\boldsymbol{l},\boldsymbol{r})} W^{\delta-\delta(\boldsymbol{l},\boldsymbol{r})}_{\beta(d^t,\boldsymbol{r}-\boldsymbol{l})}$$

where the summation indices are as described above. What this means is that with special care, one can be able to show that the generating function for the refined Severi degrees on general toric surface defined by a h-transverse lattice polygon does satisfy similar properties as in Theorem 3.1.

3.2. Relation to the Generating Functions of the Refined Invariants

The refined invariants $\widetilde{N}^{(S,L),\delta}(y)$ introduced in [**GS14**] are symmetric Laurent polynomials in a variable y whose coefficients can be expressed universally as polynomials in the four intersection numbers L^2, LK_S, K_S and $c_2(S)$ on the surface. It is conjectured [**GS14**, Conj. 62] that the generating function for the refined invariants is multiplicative. In this section we state an explicit version of this conjecture and prove some partial results towards the conjecture for $\mathbb{P}^2$ and rational ruled surfaces. For toric surfaces S and sufficiently ample line bundles L, the refined invariants $\widetilde{N}^{(S,L),\delta}(y)$ and the refined Severi degrees are conjectured to agree.

Conjecture 3.25. [**GS14**, Conj. 80]. *Let (S, L) be a pair of a smooth toric surface and a line bundle on L.*

(1) If L is δ-very ample on S, then $\widetilde{N}^{(S,L),\delta}(y) = N^{(S,L),\delta}(y)$.

(2) $\widetilde{N}^{d,\delta}(y) = N^{d,\delta}(y)$ for $\delta \leq 2d - 2$.

(3) $\widetilde{N}^{(\mathbb{P}^1\times\mathbb{P}^1,dH+cF),\delta}(y) = N^{(\mathbb{P}^1\times\mathbb{P}^1,dH+cF),\delta}(y)$ *for* $\delta \le \min(2d,2c)$.

(4) $\widetilde{N}^{(\Sigma_m,dH+cF),\delta}(y) = N^{(\Sigma_m,dH+cF),\delta}(y)$ *for* $\delta \le \min(2d,c)$.

REMARK 3.26. If Conjecture 3.25 above is true then it implies that the refined Severi degrees for (S,L) can also be expressed universally as polynomials in the intersection numbers of the pair. Explicitly, it means that for every $\delta \ge 0$ then the conjectural (Conjecture 1.26) universal polynomials $T_\delta \in \mathbb{Q}[y^{\pm 1}][q,r,s,t]$ satisfy

$$N^{(S,L),\delta}(y) = T_\delta(L^2, LK_S, K_S, c_2(S)).$$

This implies that $N_\delta(S,L;y) = \widetilde{N}^{(S,L),\delta}(y)$ for any pair of a toric surface and a line bundle (S,L). In particular $N_\delta(d;y) = \widetilde{N}^{d,\delta}(y)$ for all d, δ, and $N_\delta((\Sigma_m, cF + dH);y) = \widetilde{N}^{(\Sigma_m,dH+cF),\delta}(y)$ for all m,d,c,δ.

We now state the explicit conjectural generating function for the $N_\delta(S,L;y)$. We have already introduced (in §1.3 of Chapter 1) the Eisenstein series $G_k(q)$, the discriminant $\Delta(q)$ and their modifications $\widetilde{\Delta}(y,q), \widetilde{DG}_2(y,q)$. For purely spacial constraints we shall often write $\widetilde{\Delta}, \widetilde{DG}_2, B_i$ for the power series $\widetilde{\Delta}(y,q), \widetilde{DG}_2(y,q), B_i(y,q)$ respectively. We first restate the conjectural generating function for the refined invariants and give two equivalent reformulations of the conjecture.

CONJECTURE 3.27. *([**GS14**]) There exist universal power series B_1, B_2 in $\mathbb{Q}[y,y^{-1}][\![q]\!]$, such that for all pairs (S,L) of a smooth projective surface S and a line bundle L on S, we have*

$$\sum_{\delta\ge 0} \widetilde{N}^{(S,L),\delta}(y)(\widetilde{DG}_2)^\delta = \frac{(\widetilde{DG}_2/q)^{\chi(L)} B_1^{K_S^2} B_2^{LK_S}}{(\widetilde{\Delta}\cdot D\widetilde{DG}_2/q^2)^{\chi(\mathcal{O}_S)/2}} \tag{3.20}$$

where as before, $D := q\frac{\partial}{\partial q}$.

We now give two equivalent reformulations. Note that $\widetilde{DG}_2$ as a power series in q starts with q (see (1.39) above), let

$$g(t) := g(y,t) = t + ((-y^2-4y-1)/y)t^2 + ((y^4+14y^3+30y^2+14y+1)/y^2)t^3 + O(t^4)$$

be its compositional inverse i.e. $g(t)$ is a power series such that $\widetilde{DG}_2(y,g(t)) = t$ and conversely $g(y,\widetilde{DG}_2(y,q)) = q$. Write $g'(t) := \frac{\partial g}{\partial t}$.

REMARK 3.28. Let $R \in \mathbb{Q}[y^{\pm 1}][[q]]$ be a formal power series. For polynomials $M_\delta((S,L);y) \in \mathbb{Q}[y^{\pm 1}]$ the following three formulas are equivalent:

(1) $\displaystyle\sum_{\delta\ge 0} M_\delta((S,L);y)(\widetilde{DG}_2)^\delta = \frac{(\widetilde{DG}_2/q)^{\chi(L)}B_1^{K_S^2}B_2^{LK_S}}{(\widetilde{\Delta}\cdot D\widetilde{DG}_2/q^2)^{\chi(\mathcal{O}_S)/2}}R.$

(2) $\displaystyle\sum_{\delta\ge 0} M_\delta((S,L);y)t^\delta = \frac{(t/g(t))^{\chi(L)}B_1(y,g(t))^{K_S^2}}{B_2(y,g(t))^{-LK_S}}\left(\frac{g(t)g'(t)}{\widetilde{\Delta}(y,g(t))}\right)^{\chi(\mathcal{O}_S)/2} R(y,g(t)).$

(3) For all $\delta\ge 0$

$$M_\delta((S,L);y) = \operatorname*{Coeff}_{q^{(L^2-LK_S)/2}}\left[\widetilde{DG}_2^{\chi(L)-1-\delta}\frac{B_1^{K_S^2}B_2^{LK_S}D\widetilde{DG}_2}{(\widetilde{\Delta}\cdot D\widetilde{DG}_2)^{\chi(\mathcal{O}_S)/2}}R\right].$$

Proof. (2) is equivalent to (1) by noting that

$$D\widetilde{DG}_2(y,g(t)) = \frac{g(t)}{g'(t)}\frac{\partial\widetilde{DG}_2(y,g(t))}{\partial t} = \frac{g(t)}{g'(t)}.$$

Let A be a commutative ring, and let $f\in A[\![q]\!]$, $h\in q+q^2A[\![q]\!]$. Then we get by the residue formula that

$$f(q) = \sum_{l=0}^\infty h(q)^l \operatorname*{Coeff}_{q^0}\left[\frac{f(q)Dh(q)}{h(q)^{l+1}}\right].$$

Applying this with $h(q)=\widetilde{DG}_2$, and using the equality $\chi(L)=\frac{1}{2}(L^2-LK_S)+\chi(\mathcal{O}_S)$, shows that (1) is equivalent to (3). $\square$

Using the Riemann-Roch theorem i.e. $\chi(L)=(L^2-LK_S)/2+\chi(\mathcal{O}_S)$ in Remark 3.28 then part (2) shows that Conjecture 3.27 is a more explicit version of Conjecture 1.28. In particular, with

$$A_1(y,t)=\left(\frac{t}{g(t)}\right)^{1/2} \text{ and } A_4(y,t)=\left(\frac{tg'(t)}{g(t)\widetilde{\Delta}(y,g(t))}\right)^{1/2}.$$

By Remark 3.26 for $\mathbb{P}^2$ and rational ruled surfaces the conjecture says in particular

$$N_\delta(d;y) = \operatorname*{Coeff}_{q^{(d^2+3d)/2}}\left[\widetilde{DG}_2^{d(d+3)/2-\delta}\frac{B_1^9}{B_2^{3d}}\left(\frac{D\widetilde{DG}_2}{\widetilde{\Delta}}\right)^{1/2}\right], \tag{3.21}$$

$$N_\delta((\Sigma_m,cF+dH);y) = \operatorname*{Coeff}_{q^{(d+1)(c+1+md/2)-1}}\left[\frac{\widetilde{DG}_2^{(d+1)(c+1+md/2)-1-\delta}B_1^8}{B_2^{2c+(m+2)d}}\left(\frac{D\widetilde{DG}_2}{\widetilde{\Delta}}\right)^{1/2}\right]. \tag{3.22}$$

Corollary 3.29. *With the the power series $B_1(y,q)$, $B_2(y,q)$ given in* [**GS14**, Conj. 67] *modulo q^{11} and in the Appendix A modulo q^{18}, we have the following*

(1) The formula (3.21) and Conjecture 3.25(2) are true for $\delta\le 17$.

(2) In case $m = 0$ the formula (3.22) and Conjecture 3.25(2) is true for $\delta \le 12$.

(3) The formula (3.22) and Conjecture 3.25(3) are true for all m and $\delta \le 8$.

PROOF. (1). Using the Caporaso-Harris recursion (Definition 1.29 and Remark 1.30), we computed the $N^{d,\delta}(y)$ for $d \le 19$, $\delta \le 19$. This also computes the $Q^{d,\delta}$ for $d \le 19$, $\delta \le 19$. Part (4) of Theorem 3.22 gives $Q^{d,\delta} = Q_\delta(d)$ for $d \ge \delta$. As $Q_\delta(d;y)$ is a polynomial of degree 2 in d, the computation above determines $Q_\delta(d;y)$ and thus the $N_\delta(y;d)$ for $\delta \le 17$, giving the claim.

(2) and (3). Using again the Caporaso-Harris recursion (Definition 1.29 and Remark 1.30), we computed the refined Severi degree $N^{(\mathbb{P}^1\times\mathbb{P}^1,cF+dH),\delta}(y)$ for $c,d \le 13, \delta \le 13$. Again this gives the $Q^{(\mathbb{P}^1\times\mathbb{P}^1,cF+dH),\delta}$ for $c,d \le 13$, $\delta \le 13$. By part (2) of Theorem 3.22 we have that $Q^{(\mathbb{P}^1\times\mathbb{P}^1,cF+dH),\delta} = Q_\delta((\mathbb{P}^1\times\mathbb{P}^1, cF+dH);y)$ for $c,d \ge \delta$. As $Q_\delta((\mathbb{P}^1\times\mathbb{P}^1, cF+dH);y)$ is a polynomial of bidegree $(1,1)$ in c,d, the computation above determines $Q_\delta((\mathbb{P}^1\times\mathbb{P}^1, cF+dH);y)$ and thus the $N_\delta((\Sigma_0, cF+dH);y)$ for $\delta \le 12$. As $Q_\delta((\Sigma_m, cF+dH);y)$ is a linear combination of 1, c, cd, m, md, md^2, in order to prove (2) we only need to determine the coefficients of m, md, md^2. For this we can restrict to the case $m = 1$, We computed $N^{(\Sigma_1,cF+dH),\delta}(y)$ for $c, \le 9$, $d \le 10$. This determines the coefficients of m, md, md^2 of $Q_\delta((\Sigma_m, cF+dH);y)$ for $\delta \le 8$, giving the claim. $\square$

As noted above, the refined Severi degrees $N^{(S,L),\delta}(y)$ specialize at $y = -1$ to the tropical Welschinger numbers $W^{(S,L),\delta}$. We specialize the above conjectures of [**GS14**] to the tropical Welschinger numbers. As the Caporaso-Harris recursion for the tropical Welschinger numbers is computationally much more efficient than that for the refined Severi degrees, the conjectures for the tropical Welschinger numbers can be proven for much higher δ. Conjecture 3.25 specializes to the following (see also [**GS14**]).

CONJECTURE 3.30. *For the stable Welschinger numbers we have*

(3.23)

$$W_\delta(d) = \operatorname*{Coeff}_{q^{(d^2+3d)/2}}\left[\overline{G}_2(q)^{d(d+3)/2-\delta}\frac{\overline{B}_1(q)^9(D\overline{G}_2(q))^{1/2}}{\overline{B}_2(q)^{3d}\eta(q)^8\eta(q^2)^2}\right],$$

(3.24)

$$W_\delta((\Sigma_m, cF+dH)) = \operatorname*{Coeff}_{q^{((d+1)(c+1+md/2)-1}}\left[\frac{\overline{G}_2(q)^{(d+1)(c+1+md/2)-1-\delta}\overline{B}_1(q)^8(D\overline{G}_2(q))^{1/2}}{\overline{B}_2(q)^{2c+(m+2)d}\eta(q)^8\eta(q^2)^2}\right]$$

where $\overline{G}_2(q), D\overline{G}_2(q), \overline{B}_1(q), \overline{B}_2(q)$ *and* $\eta(q)$ *are as in* §1.5 *of Chapter* 1.

Corollary 3.31. *With* $\overline{B}_1(q)$, $\overline{B}_2(q)$ *given below modulo* q^{31} *we have the following.*

1. *The formula* (3.23) *is true for* $\delta \leq 30$. *Furthermore for* $\delta \leq 30$ *and* $d \geq \delta/3 + 1$ *we have* $W^{d,\delta} = W_\delta(d)$.
2. *On* $\mathbb{P}^1 \times \mathbb{P}^1$ *the formula* (3.24) *is true for* $\delta \leq 20$. *Furthermore for* $\delta \leq 20$ *and* $\delta \leq \min(20, 3c, 3d)$, *we have* $W^{(\mathbb{P}^1\times\mathbb{P}^1, cF+dH),\delta} = W_\delta(\mathbb{P}_1 \times \mathbb{P}_1, cF + dH)$.
3. *For* $m > 0$, *the formula* (3.24) *is true for* $\delta \leq 11$. *Furthermore for* $\delta \leq \min(11, 3d, c)$ *we have* $W^{(\Sigma_m, cF+dH),\delta} = W_\delta(\Sigma_m, cF + dH)$.

Proof. (1) Using the Caporaso-Harris recursion (Definition 1.29, Definition 1.37 and Remark 1.30), we computed to the $W^{d,\delta}$ for $d \leq 32$, $\delta \leq 33$. This also computes the $Q^{d,\delta}(-1)$ for $d \leq 32$, $\delta \leq 33$. The same argument as in the proof of Corollary 3.29 shows (1). Using again the Caporaso-Harris recursion we computed the $W^{(\mathbb{P}^1\times\mathbb{P}^1,cF+dH),\delta}$ for $c, d \leq 21, \delta \leq 22$, and computed $W^{(\Sigma_1,cF+dH),\delta}(y)$ for $c, d, \delta \leq 13$. The same argument as in the proof of Corollary 3.29 gives (2) and (3). □

$$\begin{aligned}\overline{B}_1(q) = {}& 1 - q - q^2 - q^3 + 3q^4 + q^5 - 22q^6 + 67q^7 - 42q^8 - 319q^9 + 1207q^{10} - 1409q^{11} \\ & - 3916q^{12} + 20871q^{13} - 34984q^{14} - 37195q^{15} + 343984q^{16} - 760804q^{17} - 81881q^{18} \\ & + 5390386q^{19} - 15355174q^{20} + 8697631q^{21} + 79048885q^{22} - 293748773q^{23} \\ & + 329255395q^{24} + 1041894580q^{25} - 5367429980q^{26} + 8780479642q^{27} + 10991380947q^{28} \\ & - 93690763368q^{29} + 203324385877q^{30} + O(q^{31}),\end{aligned}$$

$$\begin{aligned}\overline{B}_2(q) = {}& 1 + q + 2q^2 - q^3 + 4q^4 + 2q^5 - 11q^6 + 24q^7 + 4q^8 - 122q^9 + 313q^{10} - 162q^{11} \\ & - 1314q^{12} + 4532q^{13} - 4746q^{14} - 13943q^{15} + 68000q^{16} - 105786q^{17} - 124968q^{18} \\ & + 1025182q^{19} - 2139668q^{20} - 443505q^{21} + 15157596q^{22} - 41007212q^{23} + 19514894q^{24} \\ & + 214218876q^{25} - 755331892q^{26} + 780656576q^{27} + 2776494907q^{28} \\ & - 13420432234q^{29} + 20749875130q^{30} + O(q^{31}).\end{aligned}$$

Conjecture 3.27 above exhibits the following conjectural principle: for the pair (S, L) of a projective surface and a line bundle, write

$$A^{(S,L)}(y,q) := \frac{B_1(y,q)^{K_S^2} B_2(y,q)^{LK_S} D\widetilde{DG}_2(y,q)}{\left(\widetilde{\Delta}(y,q) D\widetilde{DG}_2(y,q)\right)^{\chi(\mathcal{O}_S)/2}}.$$

Then for L sufficiently ample, the number of δ-nodal curves in $|L|$ that satisfy k general point conditions is

$$\operatorname*{Coeff}_{q^{(L^2-LK_S)/2}} \left[\widetilde{DG}_2(y,q)^k A^{(S,L)}(y,q)\right].$$

This principle has the following generalization. To each condition c that is imposed at points of S to curves in $|L|$ on S e.g.

- passing through points with a given multiplicity,
- S having singularity at some points

there corresponds a power series $F_c \in \mathbb{Q}[y^{\pm 1}][\![q]\!]$ satisfying the following. For L sufficiently ample, the refined count of curves in $|L|$ satisfying conditions $c_1, \ldots, c_n$ is

$$\operatorname*{Coeff}_{q^{(L^2-LK_S)/2}} \left[\prod_{i=1}^{n} F_{c_i}(y,q) A^{(S,L)}(y,q)\right].$$

Clearly under this principle, the power series corresponding to passing through general point is $\widetilde{DG}_2(y,q)$. In §3.3 we shall look at this principle in the particular case when S has some given type of singularities and later in §3.4, we shall look at this principle in the particular case where curves are required to pass through non-singular points of S with a prescribed multiplicity.

3.3. Correction Terms for Singularities

In this section we want to extend the above results and conjectures to surfaces with singularities. This is partially motivated by the paper [**LO14**], where this question is studied for the non-refined invariants for toric surfaces with *rational double points*. We have conjectured above and given evidence that there exist universal power series $A_i \in \mathbb{Q}[y^{\pm 1}][[q]]$ such that the generating functions for the refined node polynomials on smooth toric surfaces S has the form

$$\mathcal{N}(S,L;y) = A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{\chi(\mathcal{O}_S)}.$$

In light of the conjectural principle discussed in §3.2, it seems natural to conjecture that this extends to singular surfaces in the following form: for every analytic type of singularities c there is a universal power series $F_c(y,q)$ such that the generating function for a singular surface S has the form

$$\mathcal{N}(S,L;y) = A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{\chi(\mathcal{O}_S)} \prod_c F_c^{n_c},$$

where n_c is the number of singularities of S of type c. For the case of toric surfaces given by h-transversal lattice polygons with only rational double points this problem has been solved in [**LO14**] for the (non-refined) Severi degrees.

3.3.1. Basic Properties of Surface Singularities. The main objects in the study of surface singularities consist of a (possibly) singular surface S and a *resolution* $f : X \to S$. Recall that a resolution of an isolated singularity (S,p) is, by definition, a projective morphism $\pi : X \to S$ where X is smooth, which induces an isomorphism $X - \pi^{-1}(p) \to S - \{p\}$. In particular, π is birational. In most scenarios, the goal is to use a resolution $X \to S$ to quantify the difference between S and X by associating invariants to the singularities. In this section, we study the difference between S and X by comparing the generation functions of the refined node polynomials. We first recall some basic facts about singularities on complex algebraic surfaces and their resolutions.

Consider the affine variety V defined as the zero locus of the polynomial $f = x^2+y^2+z^{n+1}$ for $n \geq 1$. An elementary calculation shows that the point $p = (0,0,0)$ is an isolated singularity of V. This singularity belongs to a prominent class of singularities on complex surfaces - the *rational double points*, also referred to as the *Du Val singularities*. They come in three major classes A_n, D_n(for $n \geq 4$) and E_n(for $n = 6,7,8$) as the isolated singularities of the hypersurfaces defined by the polynomials in Table 1.

The rational double points also occur as *quotient singularities* i.e. they are isomorphic to $\mathbb{C}^2/G$ where G is a finite subgroup of $SU(2,\mathbb{C})$. In general, quotient singularities are defined as follows. Let V be a germ of a point $p \in X$ and G

Type	Polynomial	Group
A_n	$x^2 + y^2 + z^{n+1}$	cyclic
D_n	$x^2 + y^2 z + z^{n-1}$	binary dihedral
E_6	$x^2 + y^3 + z^4$	binary tetrahedral
E_7	$x^2 + y^3 + yz^3$	binary octahedral
E_8	$x^2 + y^3 + z^5$	binary icosahedral

TABLE 1. The Du Val singularities.

be a finite group of analytic/algebraic automorphisms of V acting freely on $V - \{p\}$. By a theorem of Cartan [**Car57**], the quotient space V/G has the structure of a normal two dimensional complex analytic/algebraic space with an isolated singularity. Furthermore, the projection map $V \to V/G$ is analytic/algebraic.

EXAMPLE 3.32 (Cyclic quotient singularity of type $\frac{1}{r}(1, a)$). Let $G = \mathbb{Z}/r$ be the cyclic group of order r generated by

$$g = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{pmatrix} \in GL(2, \mathbb{C})$$

where ε is a primitive rth root of unity, and where a is coprime to r. The affine variety $\mathbb{C}^2$ admits a standard action by G i.e. the restriction of the standard action on $\mathbb{C}^2$ by $GL(2, \mathbb{C})$. A point on $X := \mathbb{C}^2/G$ is an orbit of G on $\mathbb{C}^2$ and thus the (orbit of the) origin is an isolated singularity of the quotient X. The coordinate ring $\mathbb{C}[X]$ is the ring of invariants $\mathbb{C}[x, y]^G$ of the induced action of G on $\mathbb{C}[x, y]$. This is the action

$$G \times \mathbb{C}[x, y] \to \mathbb{C}[x, y]$$
$$(g, F) \mapsto g(F)$$

where $g(F)$ it the polynomial function such that $g(F)(p) = F(g(p))$. This action maps a monomial $x^m y^n$ to $\varepsilon^{m+an} x^m y^n$ and thus a monomial in invariant under this action if and only if $m + an \equiv 0 \bmod r$. To see this, write $\overline{M} \cong \mathbb{Z}^2$ for the lattice of Laurent monomials in x, y and $\overline{N}$ for the dual lattice with basis e_1, e_2. Consider the lattice

$$N = \overline{N} + \mathbb{Z} \cdot \frac{1}{r}(1, a) \supset \overline{N}$$

and let $M \subset \overline{M}$ denote the dual sublattice. A Laurent monomial lies in M if and only if the pairing $\langle \frac{1}{r}(1,a), x^m y^n \rangle = \frac{m+an}{r}$ is integral. This means that $x^m y^n \in M$ if and only if $m + an \equiv 0 \bmod r$.

REMARK 3.33. Let G be a non-trival finite subgroup of $SU(2)$ as listed in Table 1. Then by [**Dur79**, Prop. 5.2.], $\mathbb{C}^2/G$ is isomorphic to $f^{-1}(0)$ where f is the corresponding polynomial in column 3. In other words, any ordinary double point on an algebraic surface can be realized locally as a quotient singularity. A cyclic quotient singularity of type $\frac{1}{r}(1,a)$ is a rational double point if in particular, $a+1 \equiv 0 \bmod r$. General information about rational double points can be found in [**Dur79, DPT80**]. Furthermore, there exist various generalizations of the classification of rational double points, for instance, to rational triple points [**Art66**], to elliptic singularities [**Wag70**], and to minimally elliptic singularities [**Lau77**].

It is known, due to Zariski and Abhyankar [**Lip69**, §II], that any surface singularity has a resolution. Furthermore, it is also known that any birational morphism on projective surfaces can be factored into a finite sequence of monoidal transformations(blow up at points) and their inverses [**Har77**, Thm. V.5.5]. Thus the monoidal transformation is fundamental to the study of resolution of singularities on surfaces. In particular, we have the following important theorem relating the intersection product on the Picard group $\mathrm{Pic}(S)$ on a surface S to the intersection product on $\mathrm{Pic}(\widetilde{S})$ where $\widetilde{S}$ is obtained from S by blowing up a point $p \in S$.

THEOREM 3.34. [**Bea96**, Prop. II.3]. *Let $\pi : \widetilde{S} \to S$ be a blow up of a point p of a surface S and let E be the exceptional divisor i.e. $E := \pi^{-1}(p)$.*

(1) There is an isomorphism $\mathrm{Pic}(S) \oplus \mathbb{Z} \xrightarrow{\sim} \mathrm{Pic}(\widetilde{S})$ defined by $(D, n) \mapsto \pi^* D + nE$.
(2) Let D, D' be divisors on S then $\pi^* D \cdot \pi^* D' = D \cdot D'$, $\pi^* D \cdot E = 0$ and $E^2 = -1$.
(3) $K_{\widetilde{S}} = \pi^* K_S + E$.

Let $\pi : X \to S$ be a resolution and denote by E the reduced preimage $\pi^{-1}(p)_{\mathrm{red}}$. Then E is called the exceptional curve of the resolution is possibly singular and reducible. We denote by E_i for $i = 1, \ldots, k$ its irreducible components. By Zariski's Main Theorem [**Har77**, Cor III.11.4] we know that E is connected. The resolution is said to be transversal if E has only ordinary double point singularities(nodal).

DEFINITION 3.35. A resolution $\pi : X \to S$ is said to be minimal any other resolution $\phi : Y \to S$ factors through π i.e. there exists a morphism $\psi : Y \to X$ such that $\phi = \pi \circ \psi$.

A curve C on a smooth surface X is called contractible if $C \simeq \mathbb{P}^1$ and $C^2 = -1$. Let $\pi : X \to S$ be a resolution and suppose that there exists a component E_i of the exceptional divisor E that is contractible. The by Castelnuovo's criterion [**Har77**, Thm. V.5.7], there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "\eta"'] & S \\ Y \arrow[r, "\pi'"'] & Z \arrow[u, "\eta'"'] \end{tikzcd}$$

where Y, Z are smooth surfaces, π' a monoidal transformation at a point $p \in Z$ and where η' a projective morphism and η an isomorphism such that $\eta(C) = \pi'^{-1}(p)$. The composition $\pi' \circ \eta$ maps E_i to a point and thus $\eta' : Z \to S$ is a resolution whose exceptional divisor has less irreducible components. Thus one can find a resolution $\pi : X \to S$ whose exceptional divisor contains no contractible curves. Minimal resolution are precisely those resolution containing no contractible curves. A theorem of Hironaka [**Hir64**], says that any normal surface singularity (S, p) admits a resolution $\pi : X \to S$. Moreover, among the resolutions of (S, p), there exists a *good resolution*: a resolution such that the exceptional locus $E = \cup E_i$ consist of smooth curves with a pair E_i, E_j meeting transversely in at most one point.

3.3.2. Refined Node Polynomials on Singular Surfaces. We now want to use the above fundamental facts to extend the conjectural principle discussed in §3.2 above. We start out by formulating a conjecture for general singular toric surfaces, and then give more precise results for specific singularities. For rational double points we conjecture that somewhat surprisingly the power series $F_c(y, q)$ is independent of y. In particular this says that the correction factor for A_n-singularities, determined in [**LO14**] for the Severi degrees, is the same for the Severi degrees and the tropical Welschinger invariants.

Now let S be a normal toric surfaces. We want to formulate a conjecture about the refined Severi degrees $N^{(S,L),\delta}(y)$. Note that the tropical curves counted in $N^{(S,L),\delta}(y)$ are not required to pass through any of the singular points of S. One can also reformulate the same conjecture in terms of the minimal resolution of S, i.e. a resolution $\pi : \widehat{S} \to S$, which contains no (-1) curves in the fibres of π.

Conjecture 3.36. *For every analytic type of singularities c there are formal power series $F_c \in \mathbb{Q}[y^{\pm 1}][[q]]$, $\widehat{F}_c \in \mathbb{Q}[y^{\pm 1}][[q]]$ such that the following hold. Let (S, L) be a pair of a projective toric surface and a toric line bundle on S. Let $\widehat{S}$ be a minimal toric resolution of S and denote by L also the pullback of L to $\widehat{S}$. Define $N^{(\widehat{S},L),\delta}(y) := N^{(S,L),\delta}(y)$. If L is δ-very ample on S, then*

$$N^{(S,L),\delta}(y) = \operatorname*{Coeff}_{q^{L(L-K_S)/2}} \left[\frac{\widetilde{DG}_2^{\chi(L)-1-\delta} B_1^{K_S^2}}{B_2^{-LK_S}} \left(\frac{D\widetilde{DG}_2}{\widetilde{\Delta}} \right)^{1/2} \prod_c F_c^{n_c} \right], \tag{3.25}$$

$$N^{(\widehat{S},L),\delta}(y) = \operatorname*{Coeff}_{q^{L(L-K_{\widehat{S}})/2}} \left[\frac{\widetilde{DG}_2^{\chi(L)-1-\delta} B_1^{K_{\widehat{S}}^2}}{B_2^{-LK_{\widehat{S}}}} \left(\frac{D\widetilde{DG}_2}{\widetilde{\Delta}} \right)^{1/2} \prod_c \widehat{F}_c^{n_c} \right]. \tag{3.26}$$

Here c runs through the analytic types of singularities of S, and n_c is the number of singularities of S of type c.

We can see that the two formulas formulas (3.25), (3.26) are equivalent. Note that $LK_S = LK_{\widehat{S}}$. On the other hand it is easy to see that $K_{\widehat{S}}^2 = K_S^2 - \sum_c n_c e_c$ where e_c is a rational number depending only on the singularity type c. Thus the two formulas are equivalent, via the identification

$$\widehat{F}_c(y, q) = F_c(y, q) B_1(y, q)^{e_c}.$$

It turns out that the power series $\widehat{F}_c(y, q)$ are usually simpler, so we will restrict our attention to them. Note that for a rational double point c we have $e_c = 0$ and thus $F_c = \widehat{F}_c$.

We give a slightly more precise version of the conjecture for a weighted projective space $\mathbb{P}(1, 1, m)$ and its minimal resolution Σ_m, and prove some special cases of it. In this case the exceptional divisor is the section E with self intersection $-m$. The weighted projective space $\mathbb{P}(1, 1, m)$ has one singularity of type $\frac{1}{m}(1, 1)$, i.e. the cyclic quotient of $\mathbb{C}^2$ by the m-th roots of unity μ_m acting by $\epsilon(x, y) = (\epsilon x, \epsilon y)$. We write c_m for this singularity. It is elementary to see that

$$K_{\Sigma_m}=-2H+(m-2)F=-\frac{m+2}{m}H-\frac{m-2}{m}E,\quad K_{\mathbb{P}(1,1,m)}=-\frac{m+2}{m}H,$$
$$e_{c_m}=\frac{(m-2)^2}{m},K_{\Sigma_m}^2=8,dHK_{\Sigma_m}=d(m+2),\chi(\Sigma_m,dH)=\frac{(md+2)(d+1)}{2}.$$

CONJECTURE 3.37. *If $\delta\le 2d-1$, then*

$$N^{(\Sigma_m,dH),\delta}(y)=\operatorname*{Coeff}_{q^{\frac{m}{2}d^2+(\frac{m}{2}+1)d}}\left[\frac{\widetilde{DG}_2^{\frac{m}{2}d^2+(\frac{m}{2}+1)d-\delta}B_1^8}{B_2^{d(m+2)}}\left(\frac{D\widetilde{DG}_2}{\widetilde{\Delta}}\right)^{1/2}\widehat{F}_{c_m}\right].\tag{3.27}$$

Furthermore we have for $m\ge 2$

$$\begin{aligned}\widehat{F}_{c_m}&=1-mq+\left((m-2)y+\frac{m^2+3m-10}{2}+\frac{m-2}{y}\right)q^2\\&\quad-\left((m^2+5m-14)y+\frac{m^3+9m^2+44m-132}{6}+\frac{m^2+5m-14}{y}\right)q^3\\&\quad+O(q^4),\end{aligned}$$

and

$$\begin{aligned}\widehat{F}_{c_2}&=\sum_{n\in\mathbb{Z}}(-1)^nq^{n^2}=1-2q+2q^4-2q^9+\dots,\\ \widehat{F}_{c_3}&=1-3q+(y+4+y^{-1})q^2-(10y+18+10y^{-1})q^3\\&\quad+((6y^2+70y+115+70y^{-1}+6y^{-2})q^4\\&\quad-((y^3+94y^2+473y+721y+473y^{-1}+94y^{-2}+y^{-3})q^5+O(q^6)\\ \widehat{F}_{c_4}&=1-4q+(2y+9+2y^{-1})q^2-(22y+42+22y^{-1})q^3\\&\quad+((14y^2+164y+273+164y^{-1}+14y^{-2})q^4+O(q^5).\end{aligned}$$

PROPOSITION 3.38. *Let $\delta_2=8$, $\delta_3=5$, $\delta_4=4$, $\delta_m=3$ for $m\ge5$. Then (3.27) is correct for $m\ge 2$ and $\delta\le\min(\delta_m,d)$.*

PROOF. Using the Caporaso-Harris recursion (Definition 1.29 and Remark 1.30), we computed $N^{(\Sigma_m,dH),\delta}(y)$ for $2\le m\le 4$, $\delta\le\delta_m$ and $d\le d_m$ with $d_2=10$, $d_3=7$, $d_4=6$. We find that in this range (3.27) holds for $\delta\le\min(2d-1,\delta_m)$. By part (3) of Theorem 3.22 we have that $Q^{(\Sigma_m,dH),\delta}(y)$ is a polynomial of degree 2 in d for $d\ge\delta$. By the computation we know this polynomial in the following cases: $(m=2,\ \delta\le 8)$, $(m=3,\ \delta\le5)$, $(m=4,\ \delta\le4)$. This shows the result for $m = 2,3,4$. Finally by part (5) of Theorem 3.22 we have that $Q^{(\Sigma_m,dH),\delta}(y)$ is for $d,m \geq \delta$ a polynomial in d and m of degree 2 in d and 1 in m. By the above we know this polynomial as a polynomial in d for $\delta = 0,1,2,3$ and $m = 3,4$. This determines it and thus also $Q^{(\Sigma_m,dH),\delta}(y)$ and therefore also $N^{(\Sigma_m,dH),\delta}(y)$, for $\delta = 0,1,2,3$ and $d,m \geq \delta$. The result follows. $\square$

The non-refined Severi degrees for toric surfaces with only rational double points given by h-transverse lattice polygons have been studied in [**LO14**]. The only rational double points which can occur in this case are A_n singularities. Denote by $F_{a_n}(y,q)$ the power series $F_c(y,q)$ where c is an A_n singularity. In [**LO14**] it is shown that

$$F_{a_n}(1,q) = \frac{\eta(q)^{n+1}}{\eta(q^{n+1})} = \prod_{k>0} \frac{(1-q^k)^{n+1}}{1-q^{(n+1)k}}.$$

Conjecture 3.36 above is a generalization of the results obtained there in. For A_n singularities we conjecture that the correction term $F_{a_n}(y,q)$ is independent of y. What this mean is that the generating functions for the refined node polynomials on a surface with only A_n singularities admits the same correction terms as the generating function for the (non-refined) node polynomial. This is stated precisely in Conjecture 3.39 below.

Conjecture 3.39. *Let S be projective normal toric surface with only rational double points, more precisely with n_k singularities of type A_k for all k (with n_k only nonzero for finitely many k). If L is δ-very ample on S, then*

(3.28)
$$N^{(S,L),\delta}(y) = \underset{q^{L(L-K_S)/2}}{\mathrm{Coeff}} \left[\widetilde{DG}_2^{\chi(L)-1-\delta} \frac{B_1^{K_{\widehat{S}}^2}}{B_2^{-LK_{\widehat{S}}}} \left(\frac{D\widetilde{DG}_2}{\widetilde{\Delta}}\right)^{1/2} \prod_k \left(\frac{\eta(q)^{k+1}}{\eta(q^{k+1})}\right)^{n_k} \right].$$

Remark 3.40. (1) $\mathbb{P}(1,1,2)$ has an A_1 singularity, and as we saw Σ_2 is a resolution of $\mathbb{P}(1,1,2)$. It is standard that $\theta_2(2\tau) = \frac{\eta(\tau)^2}{\eta(2\tau)}$. Thus, for $\mathbb{P}(1,1,2)$, Conjecture 3.39 is a special case of Conjecture 3.37, and Proposition 3.38 gives evidence for it.

(2) We also used a version of the Caporaso-Harris recursion for $\mathbb{P}(1,2,3)$. With the line bundle dH with d small for H the hyperplane bundle. $\mathbb{P}(1,2,3)$ has one A_1 and one A_2 singularity, also in this case Conjecture 3.39 is confirmed in the realm considered.

(3) Note that the conjecture that the $F_{a_n}(y,q)$ are independent of y says in particular that the correction factor for the A_n singularities is the same for Severi degrees and tropical Welschinger invariants.

Conjecture 3.39 can be generalized in another direction. Let S be a singular toric surface with singular points $p_1, \dots, p_r$ and a minimal toric resolution $\widehat{S}$ with exceptional divisors $E_1, \dots, E_r$. Let L be a toric line bundle on S. We have seen that $N^{(\widehat{S},L),\delta}(y) = N^{(S,L),\delta}(y)$ is a refined count of δ-nodal curves on S, which are not required to pass through the singular locus of S. In a similar way we can interpret $N^{(\widehat{S},L-k_1E_1-\ldots-k_rE_r),\delta}(y)$ as a refined count of curves in $|L|$ on S which pass through the singular points p_i with multiplicity $-k_iE_i^2$. This even makes sense if L is only a class of Weil divisors on S, the k_i are not necessarily integral but $L - k_1E_1 - \ldots - k_rE_r$ is a Cartier divisor on $\widehat{S}$. In this case the curves we count on S are Weil divisors.

Here we will consider this question only in the case that S has only A_1 singularities. Let

$$\eta(q) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \text{ and } \theta_2(q) := \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2/2}$$

be the Dirichlet eta function and one of the standard theta functions respectively. By Jacobi triple product, one can show that (see for example [**Apo76**, Thm. 14.7])

$$\eta(q^2)^3 = q^{1/4}\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)}.$$

As before, let $D = q\frac{d}{dq}$ be a differential operator. For $l \in \mathbb{Z}_{\geq 0}$, we define functions $f_l(q)$ by

(3.29)

$$f_{2k}(q) = \frac{(-1)^k}{(2k)!}\sum_{n\in\mathbb{Z}}(-1)^n\left(\prod_{i=0}^{k-1}(n^2-i^2)\right)q^{n^2} = \frac{(-1)^k}{(2k)!}\left(\prod_{i=0}^{k-1}(D-i^2)\right)\theta_2(q^2),$$

$$\begin{aligned} f_{2k+1}(q) &= \frac{(-1)^k}{(2k+1)!}\sum_{n\in\mathbb{Z}}(-1)^n(2n+1)\left(\prod_{i=0}^{k-1}\left((n+1/2)^2-(i+1/2)^2\right)\right)q^{(n+1/2)^2} \\ &= \frac{(-1)^k}{(2k+1)!}\left(\prod_{i=0}^{k-1}\left(D-(i+1/2)^2\right)\right)\eta(q^2)^3. \end{aligned}$$

In particular we have

$$f_0(q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2},$$
$$f_1(q) = \sum_{n\geq 0} (-1)^n (2n+1) q^{(n+1/2)^2},$$
$$f_2(q) = \sum_{n>0} (-1)^{n-1} n^2 q^{n^2}.$$

NOTATION 3.41. We write $N^{(S,L),\delta}_{[k_1,\ldots,k_n r]}(y) := N^{(\widehat{S},L-k_1E_1-\ldots-k_rE_r),\delta}(y)$, to stress that we view it as a count of curves on S with prescribed multiplicities at the A_1-singularities.

CONJECTURE 3.42. *Let S be a toric surface with only A_1 singularities $p_1,\ldots,p_r$. Fix $k_1,\ldots,k_r \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Let $\delta \geq 0$. Let L be a Weil divisor on S, such that $L - \sum_i k_iE_i$ is a Cartier divisor on $\widehat{S}$, which is δ-very ample on any irreducible curve in $\widehat{S}$ not contained in $E_1\cup\ldots\cup E_r$.. Then*

$$N^{(S,L),\delta}_{[k_1,\ldots,k_r]}(y) = \operatorname*{Coeff}_{q^{L(L-K_S)/2}}\left[\frac{\widetilde{DG_2}^{\chi(L)-\sum_i k_i^2-1-\delta} B_1^{K_S^2}}{B_2^{LK_S}}\left(\frac{D\widetilde{DG_2}}{\widetilde{\Delta}}\right)^{1/2}\prod_{i=1}^r f_{2k_i}(q)\right]. \tag{3.30}$$

Thus we claim that the correction factors for points of multiplicity k at A_1 singularities of S are given by the quasimodular forms $f_k(q)$. Equivalently we can look at the same question on the blowup $\widehat{S}$. Write $\widehat{L} := L - k_1E_1 - \ldots - k_rE_r$ and

$$\overline{f}_k(q) = \frac{f_k(q)}{q^{k^2/4}}, \quad k \in \mathbb{Z}_{\geq 0},$$

then (with the same assumptions) (3.30) is clearly equivalent to

$$N^{(\widehat{S},\widehat{L}),\delta}(y) = \operatorname*{Coeff}_{q^{\widehat{L}(\widehat{L}-K_{\widehat{S}})/2}}\left[\frac{\widetilde{DG_2}^{\chi(\widehat{L})-1-\delta} B_1^{K_{\widehat{S}}^2}}{B_2^{\widehat{L}K_{\widehat{S}}}}\left(\frac{D\widetilde{DG_2}}{\widetilde{\Delta}}\right)^{1/2}\prod_{i=1}^r \overline{f}_{2k_i}(q)\right]. \tag{3.31}$$

In other words, the correction factors for $\widehat{L}$ not being sufficiently ample on $\widehat{S}$ are the $\overline{f}_l(q)$.

REMARK 3.43. Under the assumptions of the conjecture, if the k_i are sufficiently large with respect to δ, then $\widehat{L}$ will be δ-very ample on $\widehat{S}$. This means by Conjecture 3.27 that for large l the correction factor $\overline{f}_l(q)$ should be 1 modulo some high power of q. In fact we find the following. For $l \in \mathbb{Z}_{>0}$ we can rewrite

$$\overline{f}_l(q) = \sum_{m \geq 0} (-1)^m \frac{2m+l}{m+l} \binom{m+l}{l} q^{m(m+l)}.$$

In particular $\overline{f}_l(q) \equiv 1 \mod q^{l+1}$.

Proof. First we deal with the case l even. Note that

$$\prod_{i=0}^{k-1}(n^2 - i^2) = n \prod_{i=-k-1}^{k-1}(n-i).$$

Thus we get for $k > 0$

$$\begin{aligned}\overline{f}_{2k}(q) &= \frac{(-1)^k}{(2k)!} \sum_{n \in \mathbb{Z}} (-1)^n \prod_{i=0}^{k-1}(n^2 - i^2) q^{n^2-k^2} \\ &= \sum_{n \geq k} (-1)^{n-k} \frac{2n}{2k} \binom{n+k-1}{2k-1} q^{n^2-k^2},\end{aligned}$$

where we also have used that $\binom{n+k-1}{2k-1} = 0$ for $n < k$. Finally put $m = n-k$, so that

$$\frac{2n}{2k}\binom{n+k-1}{2k-1} = \frac{2m+2k}{m+2k}\binom{m+2k}{2k}$$

and $n^2 - k^2 = m(m+2k)$. The case l odd is similar. Note that

$$\prod_{i=0}^{k-1}((n+1/2)^2 - (i+1/2)^2) = \prod_{i=-k+1}^{k}(n-i).$$

Thus we get

$$\begin{aligned}\overline{f}_{2k+1}(q) &= \frac{(-1)^k}{(2k+1)!} \sum_{n \geq 0} (-1)^n (2n+1) \left(\prod_{i=0}^{k-1}((n+\frac{1}{2})^2 - (i+\frac{1}{2})^2) \right) q^{\left(n+\frac{1}{2}\right)^2 - \left(k+\frac{1}{2}\right)^2} \\ &= \sum_{n \in \mathbb{Z}} (-1)^{n-k} \frac{2n+1}{2k+2} \binom{n+k}{2k} q^{\left(n+\frac{1}{2}\right)^2 - \left(k+\frac{1}{2}\right)^2},\end{aligned}$$

and put again $m := n-k$. □

Remark 3.44. It is again remarkable that the correction factors $f_k(q)$ are independent of the variable y. In particular this means again that the correction factor is the same for the Severi degrees and for the tropical Welschinger number.

We specialise the conjecture to case that S is the weighted projective space $\mathbb{P}(1,1,2)$ with the resolution Σ_2 with more precise bounds for the validity. Note that

$\chi(\Sigma_2, dH-kE) = (d+1)^2-k^2, (dH-kE)K_{\Sigma_2} = (dH-kE)(-2H) = -4d, K_{\Sigma_2}^2 = 8.$

CONJECTURE 3.45. *Let $d,k \in \frac{1}{2}\mathbb{Z}$ with $d-k \in \mathbb{Z}$. Then for $\delta \leq 2(d-k)+1$, we have*

$$N^{(\Sigma_2, dH-kE),\delta}(y) = \underset{q^{d^2+2d-k^2}}{\mathrm{Coeff}} \left[\frac{\widetilde{DG}_2^{\,d^2+2d-k^2-\delta} B_1^8}{B_2^{4d}} \left(\frac{D\widetilde{DG}_2}{\widetilde{\Delta}} \right)^{1/2} \overline{f}_{2k}(q) \right]. \tag{3.32}$$

PROPOSITION 3.46. *(1) Conjecture 3.45 is true for all d, all $k \leq 5$ and $\delta \leq 4$.*
(2) The equation (3.32) holds for all $d, k \geq 0$ with $\delta \leq d-k$ and $\delta \leq 4$.

PROOF. We compute $N^{(\Sigma_2,dH+cF),\delta}(y) = N^{(\Sigma_2,(d+c/2)H-c/2E),\delta}(y)$ for $\delta \leq 8$, $d \leq 6$ and $c \leq 5$, using the Caporaso-Harris recursion (Definition 1.29 and Remark 1.30). We find in this realm that $N^{(\Sigma_2,(nH-kE),\delta}(y)$ is equal to the right hand side of Conjecture 3.45 for $\delta \leq 2(n-k)+1$. By Theorem 3.22 $Q^{(\Sigma_2,dH+cF),\delta}(y)$ is for fixed $c \geq 0$ and for $d \geq \delta$ a polynomial of degree 2 in d. Thus the above computations determine this polynomial for $\delta \leq 4$, and $c \leq 5$. On the other hand in dependence of c and d we have that $Q^{(\Sigma_2,dH+cF),\delta}(y)$ is for $c, d \geq \delta$ a polynomial in c and d of degree 2 in d and 1 in c. By the above we know this polynomial as a polynomial in d for $c = 4$ and $c = 5$. Thus it is determined and the claim follows. $\square$

3.4. Counting Curves With Prescribed Multiple Points

Let S be a smooth projective surface and $p_1, \ldots, p_r$ be general points on S, and let $\widehat{S}$ be the blowup of S in the p_i with exceptional divisors E_i. Let L be a line bundle on S and C be a curve in $|L|$ passing through the points p_i with multiplicity n_i for each i. The strict transform $\widetilde{C}$ of C is related to its total transform π^*C by (see [Har77, Prop. V.3.6])

$$\pi^*C = \widetilde{C} + \sum_{i=1}^{r} n_i E_i.$$

Thus the difference between the total transform and the strict transform is a collection of copies of the exceptional divisors, one copy of E_i for each time C pass through p_i. Now let L be a sufficiently ample line bundle on S, and denote by the same letter its pullback to $\widehat{S}$. Then $N^{(\widehat{S},L-\sum_i n_iE_i),\delta}(1)$ counts the complex curves on S in $|L|$ with points of multiplicity n_i in p_i which have in addition δ nodes and pass through $\dim(|L-\sum_i n_iE_i)|)-\delta$ general points of S. If L is sufficiently ample, then the multiple points at the p_i impose $\sum_i \binom{n_i+1}{2}$ independent conditions on curves in $|L|$. Furthermore we see that

$$\chi(L-\sum_i n_iE_i) = \chi(L) - \sum_i \binom{n_i+1}{2}.$$

Now assume that S is a smooth projective toric surface. Let the $p_i \in S$ be fixed points of the torus action, so that $\widehat{S}$ is again a toric surface and the exceptional divisors E_i are torus-invariant divisors. Then by the above we can view $N^{(\widehat{S},L-\sum_i n_iE_i),\delta}(y)$ as a refined count of curves in $|L|$ on S with points of multiplicity n_i at p_i for all i and in addition δ nodes which pass through

$$\dim(|L|) - \delta - \sum_i \binom{n_i+1}{2}$$

general points on S.

Notation 3.47. We denote $N^{(S,L),\delta}_{n_1,\ldots,n_r}(y) := N^{(\widehat{S},L-\sum_i n_iE_i),\delta}(y)$.

For an Eisenstein series $G_{2k}(q)$, we denote

$$\overline{G}_k(q) := G_k(q) - G_k(q^2) = \sum_{n>0} \sum_{\substack{d|n \\ \frac{n}{d} \text{ odd}}} d^{2k-1}q^n.$$

We write again $D := q\frac{\partial}{\partial q}$. Note that $D^lG_{2k}(q)$ and $D^l\overline{G}_{2k}(q)$ are quasimodular forms of weight $2k+2l$.

Conjecture 3.48. *For each $i \geq 1$ there exists a universal power series $H_i \in \mathbb{Q}[y^{\pm 1}][[q]]$, such that, whenever L be sufficiently ample with respect to δ, r and $n_1, \ldots, n_r$, we have*

(3.33)

$$N^{(S,L),\delta}_{n_1,\ldots,n_r}(y) = \operatorname*{Coeff}_{q^{(L^2-LK_S)/2}} \left[\widetilde{DG}_2^{\chi(L)-1-\delta-\sum_i \binom{n_i+1}{2}} \frac{B_1^{K_S^2} B_2^{LK_S} D\widetilde{DG}_2}{(\widetilde{\Delta}\cdot D\widetilde{DG}_2)^{\chi(\mathcal{O}_S)/2}} \prod_{i=1}^r H_{n_i} \right].$$

Furthermore we conjecture for all $m > 0$ the following:

(1) $H_m(y,q)$ can be expressed in terms of Jacobi theta functions and quasimodular forms.

(2) $H_m(1,q)$ is a (usually non-homogeneous) polynomial in the $D^lG_{2k}(q)$ of weight $\leq 4k$.

(3) $H_m(-1,q)$ is a (usually non-homogeneous) polynomial in the $D^lG_{2k}(q), D^l\overline{G}_{2k}(q)$ of weight $\leq 2k$.

For small m we explicitly conjecture the following formulas:

(1) For $m \leq 2$ we conjecture

$$H_1(y,q) = \widetilde{DG}_2(y,q), \quad H_2(y,q) = \frac{F_1(y,q)}{(y^{1/2}-y^{-1/2})^4} + \frac{F_2(y,q)}{(y^{1/2}-y^{-1/2})^2(y-y^{-1})},$$

with

$$F_1(y,q) = \sum_{n>0}\sum_{d|n}\frac{1}{2}\left(-\frac{n^3}{d^3}+\frac{n^2}{d}-\frac{n}{d}\right)(y^{d/2}-y^{-d/2})^2q^n$$
$$F_2(y,q) = \sum_{n>0}\sum_{d|n}\left(\frac{n^2}{d^2}-\frac{n}{2}\right)(y^d-y^{-d})q^n.$$

(2) For the specialisation at $y=1$ we conjecture the following (dropping the q from the notation).

$$H_1(1) = DG_2,$$
$$H_2(1) = -\frac{1}{24}DG_2 + \frac{1}{6}D^2G_2 - \frac{1}{8}DG_4 - \frac{1}{24}D^3G_2 + \frac{1}{24}D^2G_4$$
$$H_3(1) = \frac{DG_2}{90} - \frac{D^2G_2}{18} + \frac{DG_4}{24} - \frac{13D^3G_2}{288} - \frac{73D^2G_4}{1440} + \frac{DG_6}{120} - \frac{D^4G_2}{144} + \frac{13D^3G_4}{1440} - \frac{D^2G_6}{480} + \frac{D^5G_2}{2880} - \frac{D^4G_4}{2016} + \frac{D^3G_6}{6912} + \frac{\Delta}{241920}$$
$$H_4(1) = -\frac{9DG_2}{1120} + \frac{7D^2G_2}{160} - \frac{21DG_4}{640} - \frac{1063D^3G_2}{23040} + \frac{1207D^2G_4}{23040} - \frac{3DG_6}{320} + \frac{79D^4G_2}{5760} - \frac{43D^3G_4}{2304} + \frac{149D^2G_6}{26880} - \frac{DG_8}{2688} - \frac{91D^5G_2}{69120} + \frac{95D^4G_4}{48384} - \frac{461D^3G_6}{645120} + \frac{101D^2G_8}{1451520} - \frac{11\Delta}{5806080} + \frac{D^6G_2}{17280} - \frac{89D^5G_4}{967680} + \frac{D^4G_6}{25920} - \frac{D^3G_8}{207360} + \frac{D\Delta}{2903040} - \frac{D^7G_2}{967680} + \frac{D^6G_4}{580608} - \frac{D^5G_6}{1244160} + \frac{D^4G_4}{8211456} - \frac{D^2\Delta}{84913920} + \frac{\Delta G_4}{864864}$$

(3) At $y=-1$ we conjecture

$$
\begin{aligned}
H_1(-1) &= \overline{G}_2(q),\\
H_2(-1) &= \frac{1}{8}\big(\overline{G}_2 - D\overline{G}_2 + \overline{G}_4 - DG_2\big),\\
H_3(-1) &= \frac{1}{24}\overline{G}_2 - \frac{1}{24}DG_2 + \frac{7}{96}\overline{G}_4 - \frac{7}{96}D\overline{G}_2 + \frac{1}{2}\overline{G}_2^3 - \frac{1}{192}D\overline{G}_4 - \frac{5}{64}G_4\overline{G}_2 + \frac{1}{96}D^2G_2\\
&\quad - \frac{5}{1024}DG_4,\\
H_4(-1) &= \frac{3\overline{G}_2}{128} - \frac{5DG_2}{192} - \frac{67D\overline{G}_2}{1536} + \frac{67\overline{G}_4}{1536} + \frac{35D^2G_2}{2304} - \frac{247DG_4}{24576} + \frac{55\overline{G}_2^3}{144} - \frac{55G_4\overline{G}_2}{1536}\\
&\quad - \frac{11D\overline{G}_4}{4608} + \frac{D^3G_2}{192} + \frac{25D^2G_4}{6144} - \frac{7DG_6}{8192} + \frac{11\overline{G}_2^4}{8} - \frac{13\overline{G}_2D^2G_2}{192} + \frac{35\overline{G}_2DG_4}{512}\\
&\quad - \frac{21G_6\overline{G}_2}{1024} + \frac{D^2\overline{G}_4}{512}.
\end{aligned}
$$

Remark 3.49. Part (1) of Conjecture 3.48 is not formulated in a very precise way. We want to illustrate the statement for $H_1(y,q)$ and $H_2(y,q)$, which we have conjecturally determined. In addition to $D := q\frac{\partial}{\partial q}$, we also consider $' = y\frac{\partial}{\partial y}$. Writing $\widetilde{DG}_2(y,q) = \frac{F_0(y,q)}{y-2+y^{-1}}$ we have

$$
\begin{aligned}
F_0(y,q) &= -\frac{D\theta(y)}{\theta(y)} - 3G_2,\\
F_1(y,q) &= \frac{1}{2}\frac{(D\theta(y))^2}{\theta(y)^2} + 3\frac{D\theta(y)}{\theta(y)}G_2 + \frac{1}{2}\frac{D\theta(y)}{\theta(y)} + \frac{15}{8}G_4 - \frac{9}{4}DG_2 + \frac{3}{2}G_2,\\
F_2(y,q) &= -\frac{1}{2}\frac{D\theta(y)\theta'(y)}{\theta(y)^2} - \frac{1}{6}\frac{D\theta'(y)}{\theta(y)} - 2G_2\frac{\theta'(y)}{\theta(y)}.
\end{aligned}
$$

Proof. A similar computation has been done in [**GS15**, Rem 1.4]. By definition we have

$$F_0(y,q) = \sum_{m>0}\sum_{d>0} m(y^d - 2 + y^{-d})q^{md} = \sum_{md>0} my^dq^{md} - 2G_2(q) + \frac{1}{12}.$$

In [**Zag91**, page 456, compare (iii) and (vii)] it is proved that

$$\frac{\theta'(0)\theta(wy)}{\theta(w)\theta(y)} = \frac{wy-1}{(w-1)(y-1)} - \sum_{nd>0}\operatorname{sgn}(d)w^ny^dq^{nd}. \tag{3.34}$$

Write $w = e^x$ and take the coefficient of x on both sides of (3.34). By the identity [**Zag91**, eq. (7)] we have

$$\frac{x\theta'(0)}{\theta(w)} = \exp\left(2\sum_{k\geq 2} G_k(q)\frac{z_1^k}{k!}\right).$$

This gives

$$\operatorname*{Coeff}_{x}\left[\frac{\theta'(0)\theta(wy)}{\theta(w)\theta(y)}\right] = \operatorname*{Coeff}_{x^2}\left[\frac{\theta(wy)}{\theta(y)}\right] + G_2(\tau) = \frac{1}{2}\frac{\theta''(y)}{\theta(y)} + G_2(\tau) = \frac{D\theta(y)}{\theta(y)} + G_2(\tau),$$

where the last step is by the heat equation $\frac{1}{2}\theta''(y) = D\theta(y)$. On the other hand we compute

$$\operatorname*{Coeff}_{z_1}\left[\frac{wy-1}{(w-1)(y-1)} - \sum_{nd>0}\operatorname{sgn}(d)w^n y^d q^{nd}\right] = \frac{1}{12} - \sum_{nd>0} ny^d q^{nd}.$$

This proves the formula for F_0.

We have

$$F_2(y,q) = \sum_{md>0}\operatorname{sgn}(d)(m^2 - md/2)y^d)q^{md}.$$

In [**GS15**, Rem. 1.4] it is shown (the statement there contains a misprint) that

$$\sum_{md>0}\operatorname{sgn}(d)m^2y^dq^{md} = -\frac{1}{\theta(y)}\left(\frac{2}{3}D\theta'(y) + 2G_2(q)\theta'(y)\right).$$

We see by (3.34) that

$$\sum_{md>0}\operatorname{sgn}(d)(-md/2)y^d)q^{md} = \frac{1}{2}D\left(\frac{\theta'(0)\theta(wy)}{\theta(w)\theta(y)}\Big|_{w=1}\right) = \frac{1}{2}D\left(\frac{\theta'(y)}{\theta(y)}\right).$$

This shows the formula for F_2.

A similar but slightly more tedious computation shows the formula for F_1. □

The conjectural formulas of Conjecture 3.48 were found by doing computations for $\mathbb{P}^2$ and its blowup Σ_1 with exceptional divisor E. We use the Caporaso-Harris recursion formula to compute $N^{(\Sigma_1,dH+mF),\delta}(y) = N^{(\Sigma_1,(d+m)H-mcE,\delta}$ for $d\leq 11$, $m\leq 4$ and $\delta\leq 22$, in this realm the following conjecture is true.

Conjecture 3.50. *There are power series $H_m(y,q)\in\mathbb{Q}[y^{\pm 1}][[q]]$, such that the following holds. For $d>0$, and $0\leq m\leq 4$ and $\delta\leq 2d+1+m(m+1)/2$ we have*

$$N_m^{(\mathbb{P}^2,dH),\delta}(y) = \operatorname*{Coeff}_{q^{(d(d+3)/2}}\left[\widetilde{DG}_2^{\,d(d+3)/2-m(m+1)/2-\delta}\frac{B_1^9(D\widetilde{DG}_2)^{1/2}}{B_2^{-3d}\widetilde{\Delta}^{1/2}}H_m\right].$$

Furthermore $H_1(y,q)$, $H_2(y,q)$ coincide with the functions with the same name from Conjecture 3.48, and $H_i(1,q)$, $H_i(-1,q)$ coincide for $i=1,2,3,4$ with the $H_i(1)$, $H_i(-1)$ from Conjecture 3.48.

Proposition 3.51. *Conjecture 3.50 is true from $m\le 4$ and $\delta\le 9$.*

Proof. The argument is the same as in several proofs before. By Theorem 3.22 we get that $Q^{(\Sigma_1,dH+mF),\delta}$ is for $\delta\le d$ a polynomial of degree 2 in d, which we know for $9\le d\le 11$. The result follows. □

Let S be a toric surface and $\widehat{S}$ be the blowup of S in torus fixed point. Given δ, if m is sufficiently large and L is sufficiently ample on S, then $L-mE$ will be sufficiently ample on $\widehat{S}$, so that Conjecture 3.27 will apply to the pair $(\widehat{S},L-mE)$, giving

$$N_m^{(S,L),\delta}(y)=N^{(\widehat{S},L-mE),\delta}(y)=$$
$$\operatorname*{Coeff}_{q^{(L^2-LK_S)/2-\binom{m+1}{2}}}\left[\widetilde{DG}_2^{\chi(L)-1-\delta-\binom{m+1}{2}}\frac{B_1^{K_S^2-1}B_2^{LK_S+m}D\widetilde{DG}_2}{(\widetilde{\Delta}\cdot D\widetilde{DG}_2)^{\chi(\mathcal{O}_S)/2}}\right].$$

Combined with Conjecture 3.48 this leads to the following conjecture.

Conjecture 3.52. *We have*

$$\frac{H_m(y,q)}{q^{\binom{m+1}{2}}}\equiv\frac{B_2(y,q)^m}{B_1(y,q)}\mod q^{m+1}.$$

Thus, if eventually one would find a way to explicitly determine the functions $H_m(y,q)$ for all m, this could give the unknown power series $B_1(y,q)$, $B_2(y,q)$ and thus complete the conjectural formulas of [**Göt98**],[**GS14**].

It is natural to assume that the specialisation of Conjecture 3.48 and also of the previous conjectures Conjecture 3.36, Conjecture 3.42 to $y=1$ hold for the usual Severi degrees $n^{(S,L),\delta}$ for projective algebraic surfaces, not just for toric surfaces. Thus we get in particular the following generalisation of the original conjecture of [**Göt98**].

Let S be a projective algebraic surface with A_1-singularties $q_1,\ldots,q_s$. Let $p_1,\ldots p_r$ be distinct smooth points on S. Let $m_1,\ldots,m_r\in\mathbb{Z}_{>0}$, $n_1,\ldots,n_s\in\mathbb{Z}_{\ge0}$. Let $\widehat{S}$ be the blowup of S in $q_1,\ldots,q_s,p_1,\ldots p_r$ and denote E_i, F_j the exceptional divisors over q_i, p_j respectively. Let L be a $\mathbb{Q}$-Cartier Weil divisor on S, such that

$\widehat{L} := L - \sum_{i=1}^{s} m_i E_i - \sum_{i=1}^{r} n_i F_i$ is a Cartier divisor on $\widehat{S}$, which is δ-very ample on all irreducible curves in $\widehat{S}$ not contained in $E_1 \cup \ldots \cup E_s \cup F_1 \cup \ldots \cup F_r$. Denote $n^{(S,L),\delta}_{(m_1,\ldots,m_r),(n_1,\ldots,n_s)} := n^{(\widehat{S},\widehat{L}),\delta}$, which we could informally interpret as the number of curves in $|L|$ which have multiplicity m_i in p_i and n_j in q_j for all i,j and pass in addition through

$$\dim |L| - \sum_{i=1}^{r} \binom{m_i+1}{2} - \sum_{j=1}^{s} \frac{n_j^2}{4}$$

general points on S, and have δ nodes as other singularities.

Conjecture 3.53. *Under the above assumptions we have*

(3.35)

$$n^{(S,L),\delta}_{(m_1,\ldots,m_r),(n_1,\ldots,n_s)} = \operatorname*{Coeff}_{q^{(L^2-LK_S)/2}} \left[DG_2(q)^{\chi(L)-\sum_i \binom{m_i+1}{2} - \sum_j \frac{n_j^2}{4} - 1} \frac{B_1(q)^{K_S^2} B_2(q)^{LK_S} D^2G_2(q)}{(\Delta(q)\cdot D^2G_2(q))^{\chi(\mathcal{O}_S)/2}} \left(\prod_{i=1}^{r} H_{n_i}(1,q)\right)\left(\prod_{i=1}^{s} f_{m_i}(q)\right)\right].$$

APPENDIX A

The Power Series B_1 and B_2

The closed form of the power series $B_1(y,q)$ and $B_2(y,q)$ are not known explicitly. However, their first few coefficients can be computed. We list the leading terms of $B_1(y,q)$ and $B_2(y,q)$, with omitted terms determined by symmetry.

$$\begin{aligned}
B_1(y,q) &= 1 - q - (y + 3 + 1/y)q^2 + (y^2 + 10y + 17 + \ldots)q^3 - (18y^2 + 87y + 135 + \ldots)q^4 \\
&+ (12y^3 + 210y^2 + 728y + 1061 + \ldots)q^5 - (2y^4 + 259y^3 + 2102y^2 + 5952y + 8236 + \ldots)q^6 \\
&+ (162y^4 + 3606y^3 + 19668y^2 + 48317y + 64253 + \ldots)q^7 - (47y^5 + 3789y^4 + 41999y^3 + 177800y^2 \\
&+ 392361y + 505678 + \ldots)q^8 + (5y^6 + 2416y^5 + 60202y^4 + 445989y^3 + 1576410y^2 + 3197831y \\
&+ 4018919 + \ldots)q^9 - (896y^6 + 58504y^5 + 793194y^4 + 4483755y^3 + 13818256y^2 + 26192369y \\
&+ 32243357 + \ldots)q^{10} + (176y^7 + 38236y^6 + 1017512y^5 + 9382867y^4 + 43520558y^3 + 120325637y^2 \\
&+ 215688799y + 260959201 + \ldots)q^{11} - (14y^8 + 16393y^7 + 944954y^6 + 14738959y^5 + 103623419y^4 \\
&+ 412518547y^3 + 1043940859y^2 + 1785764779y + 2129062780 + \ldots)q^{12} + (4384y^8 + 631224y^7 \\
&+ 17534642y^6 + 190488676y^5 + 1092093647y^4 + 3845977628y^3 + 9041155627y^2 + 14862430058y \\
&+ 17497499443 + \ldots)q^{13} - (658y^9 + 298228y^8 + 15816382y^7 - 273455570y^6 + 2279829046y^5 \\
&+ 11131917064y^4 + 35435770399y^3 + 78257451025y^2 + 124310761787y + 144758147754 + \ldots)q^{14} \\
&+ (42y^{10} + 96604y^9 + 10758628y^8 + 308060184y^7 + 3800583626y^6 + 25834889754y^5 \\
&+ 110712006552y^4 + 323710356925y^3 + 677516096371y^2 + 1044598390812y + 1204824660925 + \ldots)q^{15} \\
&- (20284y^{10} + 5452043y^9 + 272316274y^8 + 5094738491y^7 + 48707795806y^6 + 281165238614y^5 \\
&+ 1080786159810y^4 + 2938608835049y^3 + 5869829083826y^2 + 8816117002571y + 10082791437552 + \ldots)q^{16} \\
&+ (2472y^{11} + 2015609y^{10} + 188032406y^9 + 5506997958y^8 + 75206548205y^7 + 588088410636y^6 \\
&+ 2967196356618y^5 + 10400483736235y^4 + 26552849592007y^3 + 50907878544033y^2 + 74707191955540y \\
&+ 84801344804750 + \ldots)q^{17} + O(q^{18}),
\end{aligned}$$

$$
\begin{aligned}
B_2(y,q) &= \frac{1}{(1-yq)(1-q/y)}\big(1 + 3q - (3y+1+3/y)q^2 + (y^2+8y+18+\ldots)q^3 \\
&- (13y^2+53y+76+\ldots)q^4 + (7y^3+100y^2+316y+455+\ldots)q^5 - (y^4+112y^3+779y^2 \\
&+ 2076y+2819+\ldots)q^6 + (67y^4+1243y^3+6129y^2+14386y+18870+\ldots)q^7 - (19y^5 \\
&+ 1281y^4+12417y^3+48879y^2+104034y+132579+\ldots)q^8 + (2y^6+822y^5+17542y^4 \\
&+ 117829y^3+393703y^2+775411y+965540+\ldots)q^9 - (310y^6+17206y^5+207074y^4 \\
&+ 1085712y^3+3197506y^2+5913778y+7223539+\ldots)q^{10} + (62y^7+11505y^6+267658y^5 \\
&+ 2249872y^4+9825927y^3+26163595y^2+45935572y+55208836+\ldots)q^{11} - (5y^8+5076y^7 \\
&+ 253785y^6+3555348y^5+23210920y^4+87929247y^3+215557414y^3+362229349y \\
&+ 429395117+\ldots)q^{12} + (1397y^8+174456y^7+4304488y^6+42877083y^5+231296838y^4 \\
&+ 781220881y^3+1787129788y^2+2892830316y+3388742192+\ldots)q^{13} - (215y^9+85117y^8 \\
&+ 3983060y^7+62465678y^6+484877903y^5+2249516882y^4+6909207376y^3+14901830113y^2 \\
&+ 23353834274y+27076007072+\ldots)q^{14} + (14y^{10}+28472y^9+2793096y^8+71942817y^7 \\
&+ 818536892y^6+5240193024y^5+21495922606y^4+60931593665y^3+124910088474y^2 \\
&+ 190304808803y+218642432495+\ldots)q^{15} - (6158y^{10}+1462435y^9+65354234y^8 \\
&+ 1118442331y^7+9987960061y^6+54777796045y^5+202738958803y^4+536439701989y^3 \\
&+ 1052049129591y^2+1563445962327y+1781883877192+\ldots)q^{16} + (770y^{11}+558612y^{10} \\
&+ 46524657y^9+1238412474y^8+15681201140y^7+115681622517y^6+558367283967y^5 \\
&+ 1893273288345y^4+4718572145488y^3+8899835406922y^2+12937087920811y \\
&+ 14639451592197+\ldots)q^{17} + O(q^{18})\big).
\end{aligned}
$$

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