MACHAKOS UNIVERSITY

University Examinations 2018/2019

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE

FOURTH YEAR SECOND SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS AND STATISTICS)

BACHELOR OF EDUCATION (SCIENCE)

SMA 403: GALOIS THEORY

DATE: 9/5/2019 **TIME: 2:00 – 4:00 PM**

INSTRUCTION:

Answer question ONE and any other TWO questions

QUESTION ONE (COMPULSORY) (30 MARKS)

a) Let $\mathcal{F}$ be a field. Define what is meant by an automorphism of a field $\mathcal{F}$. (2 marks)

b) Show that the set of all automorphisms of a field $\mathcal{F}$ is a group under function composition. (3 marks)

c) Describe the major concern in the study of Galois Theory hence define a Galois group. (3 marks)

d) Let $p(x) = x^4 - 7x^2 + 10$. Over the field $\mathbb{Q}$.

i) Find the zeros of the polynomial $p(x)$. (3 marks)

ii) Find the splitting field $\mathcal{F}$of the polynomial $p(x)$. (3 marks)

iii) Find the index of $\mathcal{F}$ in $\mathbb{Q}$. (3 marks)

iv) Construct the Galois group $G(\mathcal{F}/\mathbb{Q})$. (3 marks)

v) Identify all subgroups of the Galois group $G(\mathcal{F}/\mathbb{Q})$. (3 marks)

vi) $G(\mathcal{F}/\mathbb{Q})$ is isomorphic to a subgroup of S_n. Justify. (3 marks)

e) Let $\{\alpha_i : i \in \mathbb{N}\}$ be a collection of automorphism of a field E.

Show that $E(\alpha_i) = \{\alpha_i \in E: \alpha_i(a) = a \ \forall \ \alpha_i\}$ is a subfield of E, i.e. the fixed field of $E(\alpha_i)$ is a subfield of E . (4 marks)

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QUESTION TWO (20 MARKS)

a) State the fundamental theorem of Galois theory. (2 marks)

b) Consider the field $\mathbb{Q}(\sqrt{5}, \sqrt{7})$.

i) The field $\mathbb{Q}(\sqrt{5}, \sqrt{7})$ may be considered as a vector space. Identify the basis elements of this vector space and state its dimension. (4 marks)

ii) Write down the general element of the field $\mathbb{Q}(\sqrt{5}, \sqrt{7})$. (1 mark)

iii) Construct the Galois group $G(\mathbb{Q}(\sqrt{5}, \sqrt{7})/\mathbb{Q})$ by fully describing its elements. (4 marks)

iv) Identify all the subgroups of $G(\mathbb{Q}(\sqrt{5}, \sqrt{7})/\mathbb{Q})$. (3 marks)

v) Use the fundamental theorem of Galois group to show by lattice diagrams how the subgroups of $G(\mathbb{Q}(\sqrt{5}, \sqrt{7})/\mathbb{Q})$ are isomorphic to the subfields of $\mathbb{Q}(\sqrt{5}, \sqrt{7})$. (6marks)

QUESTION THREE (20 MARKS)

a) Define the following terms

i. Cyclotomic Extension. (2 marks)

ii. Normal Extension. (2 marks)

iii. Pure Radical Extension. (2 marks)

b) Let E be a field extension of F and $f(x)$ be a polynomial in $\mathcal{F}(x)$. Show that any automorphism in $G(E/\mathcal{F})$ defines a permutation of the roots of $f(x)$ that lie in E.(4 marks)

c) One zero of a polynomial $f(x) \in \mathbb{Q}(x)$ is $x = \sqrt{2} + \sqrt{11}$. Using part (b) above write down the other three zeros. (3 marks)

d) Using part (c) above find the unique monic minimal polynomial $f(x) \in \mathbb{Q}(x)$ satisfied by the four zeros. (3 marks)

e) Construct the Galois group of the polynomial in part (d) above. (3 marks)

f) Deduce the subnormal series of the Galois group in part (e) above. (1 mark)

QUESTION FOUR (20 MARKS)

a) Show that the polynomial $x^n - 1$ is solvable by radicals over $\mathbb{Q}$. (3 marks)

b) Using part (a) above determine the solvability by radicals of the polynomial $f(x) = x^5 - 1$, hence deduce its Galois group. (3marks)

c) Consider the polynomial $f(x) = x^8 - 1$ over $\mathbb{Q}$. Find the zeros of this polynomial. (4 marks)

d) Identify the splitting field of $f(x) = x^8 - 1$ over $\mathbb{Q}$. (3 marks)

e) State the Galois group of $f(x) = x^8 - 1$ over $\mathbb{Q}$ and comment on its order and nature. (2 marks)

f) Define what is meant by a cyclotomic polynomial. (2 marks)

g) Using part (c) above find $\Phi_8(x)$ the 8th cyclotomic polynomial. (3 marks)

QUESTION FIVE (20 MARKS)

a) Let F be a field of characteristic zero and let $f(x) \in \mathcal{F}[x]$ be a separable polynomial of degree n. If E is the splitting field of $f(x)$, let $\propto_1, \ldots \propto_n$ be the roots of $f(x)$ in E.

Let $\Delta = \pi_{i \neq j}(\propto_i - \propto_j)$. We define the disciminant of $f(x)$ to be Δ^2.

i) If $f(x) = ax^2 + bx + c$, show that $\Delta^2 = b^2 - 4ac$. (5 marks)

ii) $f(x) = x^3 + px + q$, show that $\Delta^2 = 4q^3 - 27q^2$. (5 marks)

b) Derive the cubic formula for the zeros of the polynomial $f(x) = ax^3 + bx^2 + cx + d$. (6 marks)

c) Find the discriminant of the cubic equation $ax^3 + bx^2 + cx + d = 0$. (4 marks)