MACHAKOS UNIVERSITY

University Examinations 2018/2019

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE

THIRD YEAR SECOND SEMESTER EXAMINATIONS FOR

BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)

BACHELOR OF SCIENCE (STATISTICS AND PROGRAMMING)

BACHELOR OF SCIENCE (MATHEMATICS)

BACHELOR OF EDUCATION

SMA 301: REAL ANALYSIS II

DATE: 23/04/2019 **TIME: 2:00 – 4:00 pm**

Answer Question One and any other two questions

QUESTION ONE (COMPULSORY) (30 MARKS)

a) Let X be a non-empty set. Explain the meaning of the following terms:

(i) a metric d on X, (3 marks)

(ii) open ball centered at a point $x \in X$, (2 marks)

(iii) a convergent sequence in a metric space (X, d). (2 marks)

b) Let $X = \mathbb{R}$ and let $d: X \to X$ be a function defined by

$$d(x, y) = |x^2 - y^2| \text{ for all } x, y \in X.$$

Show that d is NOT a metric on X. (3 marks)

c) Let d be the usual metric on R and let A be the interval $(2,5) \subset \mathbb{R}$. Show that every point of A is an interior point of A. (5 marks)

d) Let d be the usual metric on $\mathbb{R}$ and consider the function $f: (\mathbb{R}, d) \to (\mathbb{R}, d)$ defined by

$$f(x) = 5x + 3.$$

Show that f is uniformly continuous on $\mathbb{R}$. (5 marks)

e) Let (X, d) and (Y, ρ) be metric spaces and suppose that $f: (X, d) \to (Y, \rho)$ is a continuous function. Prove that if $C \subset X$ is a compact subset then $f(C)$ is compact in Y.

(5 marks)

f) Let (X, d) and (Y, ρ) be metric spaces and suppose that $f: (X, d) \to (Y, \rho)$ is a continuous function. Prove that if $U \subset Y$ is an open subset then $f^{-1}(U)$ is open in X.

(5 marks)

QUESTION TWO (20 MARKS)

a) Explain the meaning of the following terms

(i) a sequentially compact subset A of a metric space (X, d), (2 marks)

(ii) compact subset A of a metric space (X, d). (2 marks)

b) Let $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function defined by

$$d(x, y) = |e^x - e^y|.$$

Show that d is a metric on $\mathbb{R}$. (4 marks)

c) Using the Cauchy-Schwartz inequality or otherwise, show that the function $d: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined by

$$d(x, y) = \sqrt{\sum_{i=1}^{3} (x_i - y_i)^2}$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ is a metric. (10 marks)

d) Let (X, d) and (Y, ρ) be metric spaces and suppose that $f: (X, d) \to (Y, \rho)$ is a constant function. Prove that f is continuous on X. (2 marks)

QUESTION THREE (20 MARKS)

a) Let (X, d) be a metric space. Explain the meaning of the following terms

(i) limit point of a subset $A \subset X$, (2 marks)

(ii) open subset $A \subset X$, and (2 marks)

(iii) closure of a subset $A \subset X$. (2 marks)

b) Suppose that X is a non-empty set and let $d: X \times X \to \mathbb{R}$ be the discrete metric on X i.e. the metric defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise.} \end{cases}$$

Let $a \in X$ be a point. Describe the elements of the set $B(a, 2)$ and $B(a, 0.5)$. (6 marks)

c) Let (X, d) be a metric space and suppose that $U_1, U_2, \dots, U_n$ is a finite sequence of open sets. Let U be the set

$$U = \bigcap_{i=1}^{n} U_i.$$

Prove that U is an open set in X. (4 marks)

d) Let (X, d) be a metric space and let x, y, z be points in X. Show that (4 marks)

$$|d(x, y) - d(x, z)| \leq d(y, z).$$

QUESTION FOUR (20 MARKS)

a) Explain the meaning the following terms:

(i) a Cauchy sequence in a metric space (X, d), (2 marks)

(ii) uniformly equivalent metrics on a set X. (2 marks)

b) Let $d_\infty: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ where $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$.

(i) Show that d_∞ is a metric on $\mathbb{R}^2$. (4 marks)

(ii) Let a be the point $(2,2)$. Sketch the open ball $B(a, 3)$. (2 marks)

c) Let d be the Euclidean metric on $\mathbb{R}^2$. Show that d is uniformly equivalent to the metric d_∞ as defined in question 4 (b) above. (4 marks)

d) Let (X, d) be a metric space. Prove that every convergent sequence $\{x_n\}$ in X is a Cauchy sequence. Use an example to show that not every Cauchy sequence is convergent. (6 marks)

QUESTION FIVE (20 MARKS)

a) Let (X, d) be a metric space. Explain the meaning of the following terms

(i) Uniformly equivalent metrics on X. (2 marks)

(ii) The closed ball of radius r and centered at x. (2 marks)

(iii) Complete metric space (2 marks)

b) Let (X, d) be a metric space and let $\{x_n\}$ be a sequence. Show that if $\{x_n\}$ is convergent the $\{x_n\}$ is a Cauchy sequence. (3 marks)

c) Let (X, d) and (Y, ρ) be metric spaces and suppose that $f: (X, d) \rightarrow (Y, \rho)$ is a function.

(i) What does it mean to say that f is uniformly continuous on X. (3 marks)

(ii) Let d be the usual metric on $\mathbb{R}$. Show that the function $f: (\mathbb{R}, d) \rightarrow (\mathbb{R}, d)$ defined by

$$f(x) = 3x + 1$$

is uniformly continuous on $\mathbb{R}$. (5 marks)

d) Let d be the standard metric on $\mathbb{R}$ and let $A = \mathbb{Q}$ be the subset consisting of the rational numbers. Show that A is not a complete subset of $(\mathbb{R}, d)$. (3 marks)