MACHAKOS UNIVERSITY

UNIVERSITY SUPPLEMENTARY EXAMINATIONS 2018/2019

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE

FIRST YEAR SECOND SEMESTER SUPPLEMENTARY EXAMINATIONS FOR BACHELOR OF SCIENCE IN MATHEMATICS AND STATISTICS

SAC 102: MATHEMATICAL MODELLING

DATE : 25/7/2019 TIME: 2:00 - 4:00 PM

INSTRUCTIONS: Attempt Question ONE and any other TWO questions.

QUESTION ONE (30 MARKS)

a) Differentiate between Doubling time and half-life. (2 marks)

b) Find all four equilibrium solutions of the differential equation $\frac{dx}{dt} = (x^2 - 4)(x^2 + x - 12)$ and determine the stability of each equilibrium. (6 marks)

c) Find all values of a and b that satisfy $\frac{1}{(x-2)(x+5)} = \frac{a}{(x-2)} + \frac{b}{(x+5)}$ and thus calculate the integral $\int \frac{1}{(x-2)(x+5)} dx$ (5 marks)

d) Find a Particular solution to $\frac{dX}{dt} = \frac{3X+5}{2}$. Satisfying $X(0) = 7$ (4 marks)

e) Sketch the growth rate curve of the differential equation $\frac{dQ}{dt} = \sin\theta$ indicating stability . (3 marks)

f) For a Simple Epidemic Model $\frac{dp}{dt} = rp(1-p)$. Obtain the general solution, then make $P(t)$ the subject (4 marks)

g) Consider the one parameter family model described by the equation $\frac{dy}{dt} = y^3 + \alpha y + y$. Locate the bifurcation value and describe the bifurcation that takes place. (5 marks)

QUESTION TWO (20 MARKS)

a) The interaction between two species is modelled by the following pair of differential equations
$$P'(t) = 11P(t) - 15Q(t)$$
$$Q'(t) = 6P(t) - 8Q(t)$$

where $P(t)$ and $Q(t)$ give the population of the prey and predators at time t respectively.

i. Show that the characteristic equation is $k^2 - 3k + 2 = 0$. (2 marks)

ii. Find the general solution for $P(t)$ and $Q(t)$. (4 marks)

iii. Find the particular solution when $P(0) = 200$ and $Q(0) = 100$ (3 marks)

iv. Describe the long term behavior of the particular solution in (c). (2 marks)

v. Describe what happens if the initial numbers of prey and predators are equal. (1 mark)

b) Using partial fraction technique to solve

i. $\int \frac{5x-11}{x^2 - 5x + 4} dx$ (4 marks)

ii. $\int_0^2 \frac{t+1}{(t+3)(t-1)} dt$ (4 marks)

QUESTION THREE (20 MARKS)

a) For the differential equation $\frac{dp}{dt} = p^2 - 3p + 2$. Sketch:

i. The Phase diagram showing stability and (3 marks)

ii. The General Solution showing nature of stability. (3 marks)

b) Express as sums of partial fractions:

i. $\frac{-2p+4}{(p^2+1)(p+1)^2}$ (3 marks)

ii. $\frac{9}{(x-2)^3(x+1)}$ (3 marks)

c) Consider the one parameter family model described by the equation $\frac{dy}{dt} = y^3 + \alpha y + y$. Locate the bifurcation value and describe the bifurcation that takes place. (4 marks)

d) Sketch the following types of functions:

i. Increasing linear function. (1 mark)

ii. Constant function. (1 mark)

iii. Oscillating function. (1 mark)

iv. Exponentially decaying function. (1 mark)

QUESTION FOUR (20 MARKS)

a) Solve $3P^{'} = P$ when $P(0) + P(6) = 2$. (4 marks)

b) List the three ways one can determine the equilibrium and stability of a differential equation. (3 marks)

c) Consider the differential equation $\frac{dx}{dt} = 6 - 2x$. Obtain an interpretation of the general solution. (4 marks)

d) Find the approximate solution curves to $\frac{dp}{dt} = F(P) = 7(P-1)(P-3)(P-4)$.

(6 marks)

e) Draw a table that classifies the different types of Interactions between two populations $P(t)$ and $Q(t)$. (3 marks)

QUESTION FIVE (20 MARKS)

A contagious disease strikes a small country, and in response the government immediately quarantines some of the sick people. The spread of the disease is modelled by $\frac{dp}{dt} = r\left[p(t) - q(t)\right]\left[1 - v - p(t)\right]$ where t is the time in days and $P(t)$ is the fraction of the population infected at time t. the other parameters in the model are constants where r is the contagiousness, q is the fraction of the population that was quarantined, and v is the fraction of the population that was vaccinated.

a) Use separation of variables and partial fractions to find the general solution to the above differential equation. (4 marks)

b) Using the values $r = 0.03$, $v = 0.1$ and $q = 0.1$ and the initial condition $p(0) = 0.25$ find an explicit formula for $p(t)$. (4 marks)

c) Using the particular solution you found in part (b), determine the amount of time it will take for 75% of the population to become infected. (2 marks)

d) A public health official ask you to compare three separate strategies: (9 marks)

i. Double the fraction of people in quarantine which will double the value of q or

ii. Double the fraction of people vaccinated which will double the value of v, or

iii. Encourage people to avoid unnecessary contact, which will halve the value of r.

Which approach is most effective at delaying the time taken for 75% of the population to become infected? (Justify your answer). (1 mark)