MACHAKOS UNIVERSITY

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

SST402: BAYESIAN STATISTICS

DATE: DEC 2017 **TIME: 2 HOURS**

INSTRUCTIONS: Answer question ONE and any other TWO questions.

QUESTION ONE

a) Carefully outline the main stages of a typical Bayesian analysis procedure. (4 marks)

b) Define the following terms as used in Bayesian statistics:
   i. Bayes estimator.
   ii. Minimax estimator
   iii. Highest posterior density interval (HPD) (6 marks)

c) Let X be a sample of size n from a normal distribution with mean θ and variance 1. Consider estimating θ with squared error loss using two estimators:

$$\hat{\theta}(x) = 2x \quad and \quad \hat{\theta}_{MLE}(x) = x$$

Determine which of the two estimators is inadmissible. (6marks)

d) Differentiate between the following terms as used in Bayesian statistics:
   i. Confidence intervals and Credible intervals (4 marks)
   ii. Prior distribution and posterior distribution (4 marks)

e) Suppose $x_1, x_2, ..., x_n$ each have an exponential distribution with parameter θ, and suppose that the prior for θ is an exponential distribution with parameter λ. Find the posterior distribution of θ. (6 marks)

QUESTION TWO

a) Let $X_1, X_2, ..., X_n$ be i.i.d random variables having a normal distribution with unknown mean μ and known variance δ^2.Assuming a prior normal distribution for μ with mean μ_0 and variance $\delta_0^{\;2}$.Show that the posterior distribution for μ is normally distributed with mean μ_* and variance $\delta_*^{\;2}$ where;

$$\mu_* = \delta_*^{\;2}\left(\frac{\mu_0}{\delta_0^{\;2}} + \frac{n\bar{x}}{\delta^2}\right) \quad and \quad \delta_*^{\;2} = \left(\frac{1}{\delta_0^{\;2}} + \frac{n}{\delta^2}\right)^{-1}$$

(10 marks)

b) Let $X_1, X_2, ..., X_{10}$ be i.i.d random variables having a normal distribution with unknown mean θ and known variance 1.Assuming a prior normal distribution for θ with mean zero and variance 5. Let the sample mean be $\bar{x} = 1.873$

i. Compute the posterior distribution of θ (6 marks)

ii. Compute the 95% credible intervals for θ. (4 marks)

QUESTION THREE

a) Define the following terms :

i. Loss function. (2 marks)

ii. Risk function (2 marks)

b) Let $X_1, X_2, ..., X_n$ be n independent observations from a population with density function $f(x/\theta)$ where θ is unknown parameter. Let $\delta(x)$ and $\lambda(\theta)$ denote the Bayes estimator of $g(\theta)$ and prior distribution of θ respectively. By considering a quadratic loss function of the form $L(\delta(x)_, \theta) = C(\theta)[\delta(x) - g(\theta)]^2$ where $C(\theta) \geq 0$, show that $\delta(\underline{x}) = E[g(\theta)/\underline{x}]$

(7 marks)

c) Given $f(x/p) = \begin{cases} p(1-p)^{x-1}, & 0 \leq p \leq 1 \\ 0 & , \ otherwise \end{cases}$

Suppose that the prior distribution of p is;

$$h(p) = \begin{cases} 1 \ , & 0 \leq p \leq 1 \\ 0 \ , & otherwise \end{cases}$$

Find the Bayes estimator of p with respect to the loss function defined as ;

$L(\delta(x)_, p) = C(p)[\delta(x) - p]^2$ where $C(p) \geq 0$. (9 marks)

QUESTION FOUR

a) Let $X_1, X_2, ..., X_n$ be n independent observations from

$$f(x/\theta) = \begin{cases} \theta^x (1-\theta)^{1-x}, & x = 0,1 \\ 0 & , \ otherwise \end{cases}$$

Let $L(\delta(x), \theta) = [\delta(x) - \theta]^2$ .Show that the minimax estimator of θ for the squared error loss function is;

$\delta(\underline{x}) = \dfrac{\bar{x}\sqrt{n}}{1+\sqrt{n}} + \dfrac{1}{2(1+\sqrt{n})}$ where $\bar{x} = \dfrac{\sum x_i}{n}$ (15 marks)

b) Define conjugacy and explain why or why not, the beta prior is conjugate with respect to the negative Binomial likelihood. (5 marks)

QUESTION FIVE

a) Suppose that $X_1, X_2,..., X_n / \theta \approx i.i.d$ Poisson (λ).

   i. What prior is conjugate for the Poisson likelihood? Give the distribution for λ along with any associated parameters. (2 marks)

   ii. Calculate the posterior distribution of $\lambda / \underline{x}$ using your prior in (i). (4 marks)

b) The numbers of sales of a particular item from an internet retail site in each of 20 weeks are recorded. Assume that, given the value of a parameter λ, these numbers are independent observations from the Poisson (λ) distribution.

   Our prior distribution for λ is a gamma (α, β) distribution.

   i. Our prior mean and standard deviation for λ are 16 and 8 respectively. Find the values of α and β. (4 marks)

   ii. The observed numbers of sales are as follows.

   14 , 19, 14, 21, 22, 33, 15, 13, 16, 19, 27, 21, 16, 25, 14, 23, 22, 17.

   Find the posterior distribution of λ. (4 marks)

   iii. Find a 95% posterior hpd interval for λ. (2 marks)

   (Note: If $X \sim gamma\,(\alpha, \beta)$,i.e $f(x) = k\, x^{\alpha-1} e^{(-\beta x)}$, then the mean of X is $E(X) = \alpha/\beta$ and the variance of X is $\mathrm{var}(X) = \alpha/\beta^2$)

c) Discuss briefly the following Makov Chains Monte Carlo methods (MCMC) methods.

   i. Gibbs sampling. (2 marks)

   ii. Metropolis Hastings (2 marks)