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Estimation of Conditional Weighted Expected Shortfall under Adjusted Extreme Quantile Autoregression

Martin M. Kithinji[1*], Peter N. Mwita[2], Ananda O. Kube[3]

[1]Department of Mathematics, Institute for Basic Sciences, Technology and Innovation, Pan African University, Nairobi, Kenya
[2]Department of Mathematics, Machakos University, Machakos, Kenya
[3]Department of Statistics and Actuarial Sciences, Kenyatta University, Nairobi, Kenya
Email: *mkithinji@kyu.ac.ke

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Abstract

In this paper, we present an estimator that improves the well-calibrated coherent risk measure: expected shortfall by restructuring its functional form to incorporate dynamic weights on extreme conditional quantiles used in its definition. Adjusted Extreme Quantile Autoregression will is used in estimating intermediary location measures. Consistency and coherence of the estimator are also proved. The resulting estimator was found to be less conservative compared to the expected shortfall.

Keywords

Exreme Quantile Autoregression, Expected Shortfall, Value at Risk, Coherence, Risk Measurement

1. Introduction

Different regulators expect institutions under their jurisdiction to comply with the set guidelines on how to ensure capital adequacy. A major threat to capital adequacy is reserves for fluctuations in investment values due to market risk. The amount of this reserve is determined using risk measures that quantify the market risk from the downside distribution of returns. One of the basic risk measures is volatility commonly referred to as standard deviation. Popular measures of risk that have found immense use in financial risk management include Value at Risk (VaR) and Expected Shortfall (ES).

VaR is the maximum potential loss a portfolio can suffer at a certain confidence interval, say 99%, in a particular number of days called the holding period.

In addition to comparing Value at Risk and the coherent Expected Shortfall proposed in [1], it was also noted in [2] that VaR improved the "what if?" sensitivity analysis in Greeks by expressing risk in terms of currency as one number with some probability attached. This eases risk reporting and aids in comparing different portfolios. However, VaR suffers from two major shortcomings that cripple its application in risk measurement. These are: VaR is not sub-additive with respect to subportfolios as well as risk variables and the fact that it underestimates true risk by failing to appreciate the severity of losses beyond the confidence threshold, [3]. Moreover, [4] showed that it is possible to construct two portfolios with different levels of tail risk but with the same VaR.

To remedy the above shortcomings the Basel Committee on Banking Supervision during the fundamental review of the trading book in the year 2013 adopted 97.5% Expected shortfall with a horizon of one day in place of 99% VaR in quantifying market risk for banks under the Basel III framework, [5]. This decision was followed by a lot of criticism based on earlier findings in [6] that ES is not elicitable and hence impossible to backtest. The fears were quelled by [7] who derived backtest procedures for ES that do not require elicitability. In fact elicitability makes backtesting easier but is not a necessary condition for backtesting procedures, [4]. However, following findings in [8] that ES is less robust compared to VaR, any misspecification of the loss/returns distribution greatly affects the risk estimates from ES. This can be attributed to the fact that ES averages quantiles in the extreme left tail of the returns distribution (right tail of the loss distribution) with equal weights meaning that the very extreme quantiles have the same effect on the final estimate as quantiles close to VaR.

The use of equal weights was remedied in [9] using a uniformly weighted sum of a systematic sample of quantiles below the value at risk as an estimator of the expected shortfall. The number of sample quantiles used in the estimation was chosen subjectively depending on the considered sample size of returns distribution. This approach was generalized in [10] through the use of nonuniform weights in the summation of the sample quantiles below the value at risk; where the left tail of the returns distribution was considered. The weights were defined using a continuously differentiable function that assigned more weight to quantiles near α thus improving the asymptotic efficiency of the ES estimator.

This paper, presents derivation of an estimator for Weighted Expected Shortfall (WES) based on extreme quantile regression in [11] to remedy the subjectivity in the WICQF[1] estimator in [10]. To achieve this, dynamic weights are introduced in the structural form of expected shortfall. The weights depend on the distance between the extreme quantiles and the value at risk. This will allow us to use all the quantiles above VaR thus quashing the subjectivity in WICQF estimator.

2. Methodology

Consider a real valued financial time series $S_t, t \in \mathbb{R}^+ \cup \{0\}$ on a complete prob-

[1]WICQF, Weighted Conditional Integrated Quantile Function.

ability space $(\Omega, \mathcal{F}, P)$. Assume S_t is $\mathcal{F}_t$-measurable where $\{\mathcal{F}_t, t \in \mathbb{R}^+ \cup \{0\}\}$ is an increasing sequence of σ-algebras representing information available up to time *t*. In particular, let S_t be the value of a portfolio at trading time t so that the return, r_t on the portfolio at time *t* is given by

$$r_t = \frac{S_t - S_{t-\Delta t}}{S_{t-\Delta t}} \tag{1}$$

For convenience, let the corresponding loss return be given by

$$X_t = -r_t \tag{2}$$

The VaR at $(1-\alpha)\times 100\%$ of the return on this portfolio is given by the quantile at $\alpha \times 100\%$ of the loss distribution

$$q_\alpha^x = \inf\{x \in \mathbb{R} : P(X > x) \le 1-\alpha\} = \inf\{x \in \mathbb{R} : F_X(x) > \alpha\} \tag{3}$$

where $\alpha \in (0,1)$. The corresponding ES of the return on this portfolio is the expected value of the losses beyond VaR which is given by

$$ES_\alpha^x = E[X \mid X > q_\alpha^x] = \frac{1}{1-\alpha}\int_\alpha^1 q_\tau^x \mathrm{d}\tau = q_\alpha^x + \frac{1}{1-\alpha} E[X - q_\alpha^X \mid X > q_\alpha^X] \tag{4}$$

where q_α^X is the inverse of the loss distribution function $F_X(x)$

For a discrete ordered sample $x_1 \le x_2 \le \cdots \le x_n$ the corresponding VaR and ES estimates are respectively given by

$$\hat{q}_\alpha^x = x_{n\alpha} \tag{5}$$

$$\widehat{ES}_\alpha^x = \frac{1}{n - n\alpha}\sum_{i=n\alpha}^{n} x_i \tag{6}$$

where appropriate approximations are made when $n\alpha$ is not an integer.

Let us now define a conditional quantile autoregressive model on X_t of the form

$$X_t = \mu_{t,\theta} + \varepsilon_t \tag{7}$$

where $\mu_{t,\theta} \equiv f : \mathbb{R}^d \to \mathbb{R}$ is the central conditional θ-quantile of X_t and ε_t are heteroscedastic errors with zero θ-quantile. Suppose we define $\varepsilon_t = \sigma_{t,\theta} Z_t$ where $\sigma_{t,\theta} \equiv f : \mathbb{R}^d \to \mathbb{R}$ is the central conditional scale of X_t and Z_t are assumed to be iid[2] innovations with a common distribution *F*(.). To capture the ARCH effects in X_t we let $\sigma_{t,\theta}$ be a function of lagged values of ε_t, that is $|\varepsilon_{t-i}|; i \in \mathbb{N}$. This modification improves the QAR-QAR process in [11]

According to [11] the adjusted extreme conditional quantile of X_t is given by

$$\mu_{t,\theta,\alpha} = \mu_{t,\theta} + \sigma_{t,\theta}[q_\alpha^z - q_\theta^z] \tag{8}$$

defined such that if estimates of the quantiles of Z_t at α and θ are $\hat{q}_\alpha^z$ and $\hat{q}_\theta^z$ respectively then we can estimate the function by

$$\hat{\mu}_{t,\theta,\alpha} = \hat{\mu}_{t,\theta} + \hat{\sigma}_{t,\theta}[\hat{q}_\alpha^z - \hat{q}_\theta^z] \tag{9}$$

[2]Independent and identically distributed.

3. Estimation of Common Risk Measures

Using Equation (9) we obtain an estimator for the one step ahead VaR forecast as

$$\widehat{VaR}_{t+1}^{\alpha} = \hat{\mu}_{t+1,\theta,\alpha} = \hat{\mu}_{t+1,\theta} + \hat{\sigma}_{t+1,\theta}\left[\hat{q}_{\alpha}^{z} - \hat{q}_{\theta}^{z}\right] \tag{10}$$

where $\hat{\mu}_{t+1,\theta}$ and $\hat{\sigma}_{t+1,\theta}$ are the corresponding one step θ-quantile and scale estimators respectively from the linear conditional quantile process and; $\hat{q}_{\alpha}^{z}$ and $\hat{q}_{\theta}^{z}$ are obtained by inverting the overall distribution of the iid errors. Note that the overall distribution of the iid errors is obtained by splicing GPD with empirical bulk distribution at the threshold as outlined in [11] to get

$$\hat{F}(z) = 1 - \frac{m}{N}\left(1 + \frac{\hat{\lambda}(z-u)}{\hat{\beta}}\right)^{\frac{-1}{\hat{\lambda}}} \tag{11}$$

for any $\tau \in (0,1)$. $\hat{\beta}$ and $\hat{\lambda}$ are the estimated parameters of the Generalised Pareto Distribution (GPD) adopted in formulating the estimated overall distribution of the iid errors. m is the number of exceedances above a chosen threshold u from a sample of size N. Inverting $\hat{F}(z)$ we get

$$\hat{q}_{\tau}^{z} = u + \frac{\hat{\beta}}{\hat{\lambda}}\left[\left(\frac{N}{m}(1-\tau)\right)^{-\hat{\lambda}} - 1\right] \tag{12}$$

Using Equation (4) the estimator of the expected shortfall from standardized errors is given by

$$\widehat{ES}_{\alpha}(Z) = \frac{\hat{q}_{\alpha}^{z}}{1-\hat{\lambda}} + \frac{\hat{\beta} - \hat{\lambda}u}{1-\hat{\lambda}} \tag{13}$$

where $\hat{q}_{\alpha}^{z}$ is as defined in Equation (12). Similarly, using Equation (4) we have

$$\begin{aligned}\widehat{ES}_{\alpha}(X) &= \frac{1}{1-\alpha}\int_{\alpha}^{1}\left(\hat{\mu}_{t+1,\theta} + \hat{\sigma}_{t+1,\theta}\left[\hat{q}_{\tau}^{z} - \hat{q}_{\theta}^{z}\right]\right)\mathrm{d}\tau \\ &= \hat{\mu}_{t+1,\theta} + \hat{\sigma}_{t+1,\theta}\left[\widehat{ES}_{\alpha}(Z) - \hat{q}_{\theta}^{z}\right]\end{aligned} \tag{14}$$

where $\widehat{ES}_{\alpha}(Z)$ is as defined in Equation (13).

Weighted Expected Shortfall

Suppose $\alpha \in (0,1)$, then we define the Weighted Expected Shortfall for a continuous random variable X as:

$$WES_{\alpha}(X) = \int_{\alpha}^{1}\Psi_{\tau}\left(\mu_{t,\theta,\tau}\right)\mu_{t,\theta,\tau}\mathrm{d}\tau \tag{15}$$

where $\Psi_{\tau}\left(\mu_{t,\theta,\tau}\right)$, are dynamic weights (risk spectrum) that vary from one quantile to another in the integrand and α is fixed. $\mu_{t,\theta,\tau}$ is the quantile at level τ.

Assumption 1 *The dynamics of the weight function* (*risk spectrum*) *are such that for any risks X, Y*; $\alpha \in (0,1)$ *and* $\alpha \le \tau \le 1$

$$\Psi_{\tau}\left(\mu_{t,\theta,\tau}^{x}\right) \ge \Psi_{\tau}\left(\mu_{t,\theta,\tau}^{x+y}\right) \text{ and } \Psi_{\tau}\left(\mu_{t,\theta,\tau}^{y}\right) \ge \Psi_{\tau}\left(\mu_{t,\theta,\tau}^{x+y}\right) \tag{16}$$

where $\mu_{t,\theta,\tau}^{x}$ is the quantile at level τ for loss distribution of X, $\mu_{t,\theta,\tau}^{y}$ is the quantile at level τ for loss distribution of Y and $\mu_{t,\theta,\tau}^{x+y}$ is the quantile at level

τ for loss distribution of $Z = X + Y$

To reduce cumbersomeness in notations, we will drop the superscript on the quantile function, $\mu_{t,\theta,\tau}$ and reintroduce it when referring to another loss distribution different from *X*.

Proposition 1 (Weights) $\Psi_\tau(\mu_{t,\theta,\alpha})$ *defined as*

$$\Psi_\tau(\mu_{t,\theta,\alpha}) = \begin{cases} \dfrac{\exp\{-(\mu_{t,\theta,\tau} - \mu_{t,\theta,\alpha})\}}{\int_\alpha^1 \exp\{-(\mu_{t,\theta,\tau} - \mu_{t,\theta,\alpha})\}\mathrm{d}\tau} & \tau \geq \alpha \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

is an admissible risk spectrum.

Proof. We need to show that $\Psi_\tau(\mu_{t,\theta,\alpha})$ satisfies the three conditions in [12] for an admissible risk spectrum in the definition of a spectral risk measure.

1) Expanding Equation (17) we obtain

$$\begin{aligned} \Psi_\tau(\mu_{t,\theta,\tau}) &= \frac{\exp\left\{-\left(\mu_{t,\theta} + \sigma_{t,\theta}\left[q_\tau^z - q_\theta^z\right] - \left(\mu_{t,\theta} + \sigma_{t,\theta}\left[q_\alpha^z - q_\theta^z\right]\right)\right)\right\}}{\int_\alpha^1 \exp\left\{-\left(\mu_{t,\theta} + \sigma_{t,\theta}\left[q_\tau^z - q_\theta^z\right] - \left(\mu_{t,\theta} + \sigma_{t,\theta}\left[q_\alpha^z - q_\theta^z\right]\right)\right)\right\}\mathrm{d}\tau} \\ &= \frac{\exp\{-(\sigma_{t,\theta} q_\tau^z)\}}{\int_\alpha^1 \exp\{-(\sigma_{t,\theta} q_\tau^z)\} d_\tau} \geq 0 \end{aligned} \tag{18}$$

as a result of non-negativity of the exponential function. Therefore $\Psi_\tau(\mu_{t,\theta,\tau}) \geq 0$.

2) The first derivative of the weight function with respect to τ is given by

$$\begin{aligned} \frac{\partial}{\partial\tau}\Psi_\tau(\mu_{t,\theta,\tau}) &= \frac{\partial}{\partial\tau}\left[\frac{\exp\{-(\sigma_{t,\theta} q_\tau^z)\}}{\int_\alpha^1 \exp\{-(\sigma_{t,\theta} q_\tau^z)\}\mathrm{d}\tau}\right] \\ &= \frac{\left[\int_\alpha^1 \exp\{-(\sigma_{t,\theta} q_\tau^z)\}\mathrm{d}\tau\right]\left[\frac{\partial}{\partial\tau} - \sigma_{t,\theta} q_\tau^z\right]\exp\{-(\sigma_{t,\theta} q_\tau^z)\}}{\left(\int_\alpha^1 \exp\{-(\sigma_{t,\theta} q_\tau^z)\}\mathrm{d}\tau\right)^2} \\ &= \frac{-\sigma_{t,\theta}\exp\{-(\sigma_{t,\theta} q_\tau^z)\}\left[\frac{\partial}{\partial\tau} q_\tau^z\right]}{\int_\alpha^1 \exp\{-(\sigma_{t,\theta} q_\tau^z)\}\mathrm{d}\tau} \\ &= -\sigma_{t,\theta}\Psi_\tau(\mu_{t,\theta,\tau})\left[\left(\frac{N}{m}\right)^{-\lambda}(1-\tau)^{-(\lambda+1)}\right] \\ &\leq 0 \end{aligned} \tag{19}$$

for $\alpha \leq \tau < 1$ which confirms that $\Psi_\tau(\mu_{t,\theta,\tau})$ is monotonically decreasing in the interval.

3) Observe that

$$\int_\tau \Psi_\tau(\mu_{t,\theta,\tau})\mathrm{d}\tau = \int_\alpha^1 \Psi_\tau(\mu_{t,\theta,\tau})\mathrm{d}\tau = \frac{\int_\alpha^1 \exp\{-(\mu_{t,\theta,\tau} - \mu_{t,\theta,\alpha})\}\mathrm{d}\tau}{\int_\alpha^1 \exp\{-(\mu_{t,\theta,\tau} - \mu_{t,\theta,\alpha})\}\mathrm{d}\tau} = 1 \tag{20}$$

□

Replacing $\Psi_\tau\left(\mu_{t,\theta,\tau}\right)$ and $\mu_{t,\theta,\tau}$ with their respective estimates in Equation (15) we deduce the corresponding WES estimate as:

$$\begin{aligned}\widehat{WES}_\alpha(X) &= \int_\alpha^1 \hat{\Psi}_\tau\left(\hat{\mu}_{t,\theta,\tau}\right)\hat{\mu}_{t,\theta,\tau}\mathrm{d}\tau \\ &= \int_\alpha^1 \frac{\exp\left\{-\left(\hat{\mu}_{t,\theta,\tau}-\hat{\mu}_{t,\theta,\alpha}\right)\right\}}{\int_\alpha^1 \exp\left\{-\left(\hat{\mu}_{t,\theta,\tau}-\hat{\mu}_{t,\theta,\alpha}\right)\right\}\mathrm{d}\tau}\hat{\mu}_{t,\theta,\tau}\mathrm{d}\tau \\ &= \hat{\mu}_{t,\theta}+\hat{\sigma}_{t,\theta}\left[\int_\alpha^1 \frac{\hat{q}_\tau^z \exp\left\{-\left(\hat{\sigma}_{t,\theta}\hat{q}_\tau^z\right)\right\}}{\int_\alpha^1 \exp\left\{-\left(\hat{\sigma}_{t,\theta}\hat{q}_\tau^z\right)\right\}\mathrm{d}\tau}\mathrm{d}\tau-\hat{q}_\theta^z\right] \\ &= \hat{\mu}_{t,\theta}+\hat{\sigma}_{t,\theta}\left[\widehat{WES}_\alpha(Z)-\hat{q}_\theta^z\right]\end{aligned} \tag{21}$$

where $\widehat{WES}_\alpha^z$ is the estimated weighted expected shortfall of the standardized errors.

Theorem 1 (Consistency of WES estimator) $\widehat{WES}_\alpha - WES_\alpha = o_p(1)$

Proof. Observe that WES_α is a continuous function of $\mu_{t,\theta,\tau}$. Since $\hat{\mu}_{t,\theta,\tau} \xrightarrow{p} \mu_{t,\theta,\tau}$ by theorem 2 in [11] then by continuous mapping theorem in [13] we have the result. □

For a discrete ordered sample $X_1 \le X_2 \le \cdots \le X_n$, $\widehat{WES}_\alpha^d(X)$ is given by

$$\widehat{WES}_\alpha^d(X) = \sum_{i=n\alpha}^{n} \hat{\Psi}_i^d(X_i)X_i \tag{22}$$

where $X_{n\alpha}$ is as defined in Equation (6) and

$$\hat{\Psi}_i^d(X_i) = \frac{\exp -\left(X_i - X_{n\alpha}\right)}{\sum_{i=n\alpha}^{n}\exp -\left(X_i - X_{n\alpha}\right)} \tag{23}$$

Note that as indicated earlier if $n\alpha$ is not an integer then appropriate approximations are made.

4. Properties of the Weighted Expected Shortfall

In this section we look at the fundamental properties, lemmas and theorems of the loss random variable as well as the weighted expected shortfall.

Lemma 1 *For any random variable X, there exists a* $U[0,1]$ *random variable* U_X *such that* $X = F_X^{-1}(U_X)$

Proof. This has been proved through a distribution transform in [14] [Lemma 2.1]. □

Lemma 2 *For* $\alpha \in (0,1)$ *and X integrable we have*

$$WES_\alpha(X) = VaR_\alpha(X) + E\left[\Psi_\tau(X)\left(X - VaR_\alpha(X)\right)_+\right] \tag{24}$$

where $(X)_+$ means $X \mid X > 0$ and

$$\Psi_\tau(X) = \begin{cases}\dfrac{\exp -\left(X - VaR_\alpha(x)\right)}{\int_{VaR_\alpha(X)}^{\infty}\exp -\left(X - VaR_\alpha(X)\right)\mathrm{d}X} & X \ge VaR_\alpha(X) \\ 0 & \text{otherwise}\end{cases}$$

Proof. By definition

$$\begin{aligned} WES_\alpha(X) &= \int_\alpha^1 \Psi_\tau(\mu_{t,\theta,\tau})\mu_{t,\theta,\tau}\mathrm{d}\tau = \int_\alpha^1 \Psi_\tau(\mu_{t,\theta,\tau})F_X^{-1}(\tau)\mathrm{d}\tau \\ &= \int_\alpha^1 \Psi_\tau(\mu_{t,\theta,\tau})\left[F_X^{-1}(\alpha)+F_X^{-1}(\tau)-F_X^{-1}(\alpha)\right]\mathrm{d}\tau \\ &= F_X^{-1}(\alpha)\int_\alpha^1 \Psi_\tau(\mu_{t,\theta,\tau})\mathrm{d}\tau + \int_\alpha^1 \Psi_\tau(\mu_{t,\theta,\tau})\left[F_X^{-1}(\tau)-F_X^{-1}(\alpha)\right]\mathrm{d}\tau \\ &= VaR_\alpha(X)+E\left[\Psi_\tau(U_X)\left(F_X^{-1}(U_X)-VaR_\alpha(X)\right)_+\right] \\ &= VaR_\alpha(X)+E\left[\Psi_\tau(X)\left(X-VaR_\alpha(X)\right)_+\right] \end{aligned} \quad (25)$$

where

$$\Psi_\tau(U_X)=\begin{cases} \dfrac{\exp-\left(F_X^{-1}(U_X)-VaR_\alpha(x)\right)}{\int_\alpha^1 \exp-\left(F_X^{-1}(U_X)-VaR_\alpha(x)\right)\mathrm{d}U_X} & F_X^{-1}(U_X)\ge VaR_\alpha(x) \\ 0 & \text{otherwise} \end{cases} \quad (26)$$

□

Lemma 3 *For* $\alpha\in(0,1)$ *and X integrable we have*

$$\begin{aligned} WES_\alpha(X) &= E\left[\Psi_\tau(X)XI_{X>VaR_\alpha(X)}\right] \\ &\quad +\Psi_\tau\left(VaR_\alpha(X)\right)VaR_\alpha(X)\left[Pr\left(X\le VaR_\alpha(X)\right)-\alpha\right] \end{aligned} \quad (27)$$

Proof. From Lemma 2 we have

$$\begin{aligned} WES_\alpha(X) &= VaR_\alpha(X)+E\left[\Psi_\tau(X)\left(X-VaR_\alpha(X)\right)I_{X>VaR_\alpha(X)}\right] \\ &= E\left[\Psi_\tau(X)XI_{X>VaR_\alpha(X)}\right]+VaR_\alpha(X)\left\{1-E\left[\Psi_\tau(X)I_{X>VaR_\alpha(X)}\right]\right\} \\ &= E\left[\Psi_\tau(X)XI_{X>VaR_\alpha(X)}\right] \\ &\quad +VaR_\alpha(X)\left\{1-\left[1-\Psi_\tau\left(VaR_\alpha(X)\right)Pr\left(X=VaR_\alpha(X)\right)\right]\right\} \\ &= E\left[\Psi_\tau(X)XI_{X>VaR_\alpha(X)}\right] \\ &\quad +\Psi_\tau\left(VaR_\alpha(X)\right)VaR_\alpha(X)\left[Pr\left(X\le VaR_\alpha(X)\right)-\alpha\right] \end{aligned} \quad (28)$$

□

Definition 1 *Let X be a bounded, integrable random variable. For* $\alpha\in(0,1)$ *and* $x\in\mathbb{R}$ *we define the generalized indicator function*

$$I_{X\ge x}^{(\alpha)}=\begin{cases} I_{X>x}, & \text{if } Pr(X=x)=0 \\ I_{X>x}+\dfrac{Pr(X\le x)-\alpha}{Pr(X=x)}I_{X=x}, & \text{if } Pr(X=x)\ne 0 \end{cases} \quad (29)$$

Lemma 4 *Let X be a bounded, integrable random variable. For* $\alpha\in(0,1)$ *and* $x\in\mathbb{R}$, *the following holds*

1) $0\le I_{X\ge VaR_\alpha(X)}^{(\alpha)}\le 1$

2) $E\left[I_{X\ge VaR_\alpha(X)}^{(\alpha)}\right]=1-\alpha$

3) $E\left[\Psi_\tau(X)I_{X\ge VaR_\alpha(X)}^{(\alpha)}\right]=1$

4) $E\left[\Psi_\tau(X)XI^{(\alpha)}_{X\geq VaR_\alpha(X)}\right]=WES_\alpha(X)$

Proof. For proof of 1) and 2) see Lemma 3.5 in [14].

Prove of 3) follows trivially from admissibility of $\Psi_\tau(X)$ in Proposition 1.

To prove 4) we note that if $Pr(X=VaR_\alpha(X))=0$, then $Pr(X\leq VaR_\alpha(X))=\alpha$ hence by lemma 3 we have

$$E\left[\Psi_\tau(X)XI^{(\alpha)}_{X\geq VaR_\alpha(X)}\right]=E\left[\Psi_\tau(X)XI_{X>VaR_\alpha(X)}\right]=WES_\alpha(X) \tag{30}$$

If $Pr(X=VaR_\alpha(X))>0$ then

$$\begin{aligned}
&E\left[\Psi_\tau(X)XI^{(\alpha)}_{X\geq VaR_\alpha(X)}\right]\\
&=E\left[\Psi_\tau(X)XI_{X>VaR_\alpha(X)}\right]\\
&\quad+\frac{Pr(X\leq VaR_\alpha(X))-\alpha}{Pr(X=VaR_\alpha(X))}E\left[\Psi_\tau(X)XI_{X=VaR_\alpha(X)}\right]\\
&=E\left[\Psi_\tau(X)XI_{X>VaR_\alpha(X)}\right]\\
&\quad+\frac{Pr(X\leq VaR_\alpha(X))-\alpha}{Pr(X=VaR_\alpha(X))}\Psi_\tau(VaR_\alpha(X))VaR_\alpha Pr(X=VaR_\alpha(X))\\
&=E\left[\Psi_\tau(X)XI_{X>VaR_\alpha(X)}\right]+\left[Pr(X\leq VaR_\alpha(X))-\alpha\right]\Psi_\tau(VaR_\alpha(X))VaR_\alpha(X)\\
&=WES_\alpha(X)
\end{aligned} \tag{31}$$

□

Theorem 2 (Subadditivity of WES) *Given that* $\Psi_\tau(X)$ *satisfies assumption* 1 *then* $WES_\alpha(X)$ *is subadditive. That is*

$$WES_\alpha(X+Y)\leq WES_\alpha(X)+WES_\alpha(Y) \tag{32}$$

Proof. By **lemma 4** 4) we have,

$$\begin{aligned}
&WES_\alpha(X)+WES_\alpha(Y)-WES_\alpha(X+Y)\\
&=E\left[\Psi_\tau(X)XI^{(\alpha)}_{X\geq VaR_\alpha(X)}\right]+E\left[\Psi_\tau(Y)YI^{(\alpha)}_{Y\geq VaR_\alpha(Y)}\right]\\
&\quad-E\left[\Psi_\tau(X+Y)(X+Y)I^{(\alpha)}_{X+Y\geq VaR_\alpha(X+Y)}\right]\\
&=E\left[X\left(\Psi_\tau(X)I^{(\alpha)}_{X\geq VaR_\alpha(X)}-\Psi_\tau(X+Y)I^{(\alpha)}_{X+Y\geq VaR_\alpha(X+Y)}\right)\right]\\
&\quad+E\left[Y\left(\Psi_\tau(Y)I^{(\alpha)}_{Y\geq VaR_\alpha(Y)}-\Psi_\tau(X+Y)I^{(\alpha)}_{X+Y\geq VaR_\alpha(X+Y)}\right)\right]
\end{aligned} \tag{33}$$

Let

$$M=(X-VaR_\alpha(X))\left(\Psi_\tau(X)I^{(\alpha)}_{X\geq VaR_\alpha(X)}-\Psi_\tau(X+Y)I^{(\alpha)}_{X+Y\geq VaR_\alpha(X+Y)}\right)$$

We show that $E[M]\geq 0$. Note that $0\leq I^{(\alpha)}_{X+Y\geq VaR_\alpha(X+Y)}\leq 1$ by **lemma 4** 1). When $X>VaR_\alpha(X)$ then $M\geq 0$ since the highest value $I^{(\alpha)}_{X+Y\geq VaR_\alpha(X+Y)}$ can attain is 1 at which point $\Psi_\tau(X)-\Psi_\tau(X+Y)\geq 0$. When $X=VaR_\alpha(X)$, then $M=0$. Finally, if $X<VaR_\alpha(X)$ then $M\geq 0$ since $\Psi_\tau(X+Y)I^{(\alpha)}_{X+Y\geq VaR_\alpha(X+Y)}\geq 0$. Therefore

$$
\begin{aligned}
&E[M] \\
&= E\left[\left(X - VaR_\alpha(X)\right)\left(\Psi_\tau(X) I^{(\alpha)}_{X \geq VaR_\alpha(X)} - \Psi_\tau(X+Y) I^{(\alpha)}_{X+Y \geq VaR_\alpha(X+Y)}\right)\right] \\
&\geq 0
\end{aligned}
$$

By **Lemma 4** 3)

$$
\begin{aligned}
&E\left[VaR_\alpha(X)\left(\Psi_\tau(X) I^{(\alpha)}_{X \geq VaR_\alpha(X)} - \Psi_\tau(X+Y) I^{(\alpha)}_{X+Y \geq VaR_\alpha(X+Y)}\right)\right] \\
&= 0 \qquad \text{by proposition 1}
\end{aligned}
$$

and so we obtain

$$E\left[X\left(\Psi_\tau(X) I^{(\alpha)}_{X \geq VaR_\alpha(X)} - \Psi_\tau(X+Y) I^{(\alpha)}_{X+Y \geq VaR_\alpha(X+Y)}\right)\right] \geq 0$$

Similarly, it can be shown that

$$E\left[Y\left(\Psi_\tau(Y) I^{(\alpha)}_{Y \geq VaR_\alpha(Y)} - \Psi_\tau(X+Y) I^{(\alpha)}_{X+Y \geq VaR_\alpha(X+Y)}\right)\right] \geq 0$$

Therefore

$$WES_\alpha(X) + WES_\alpha(Y) - WES_\alpha(X+Y) \geq 0$$

□

Remark Note that both assumption 1 and theorem 2 can be generalized for any number of losses.

Theorem 3 (Monotonicity of WES) *For* $\alpha \in (0,1)$ $WES_\alpha(X)$ *is monotonic. That is if* $X \leq Y$ always *then*

$$WES_\alpha(X) \leq WES_\alpha(Y)$$

Proof. The result follows from Proposition 3.5(i) in [15] and monotonicity of both the quantile function and the Lebesgue integral.

Theorem 4 (Coherence of WES) *Given that* $\Psi_\tau(X)$ *satisfies assumption* 1 *then* $WES_\alpha(X)$ *is coherent.*

Proof. We show that $WES_\alpha(X)$ satisfies the four axioms of definition 2.7 in [4]. We note that the proves of axiom 1) and 3) are trivial. Axiom 2) and 4) follows from theorems 2 and 3 respectively.

5. Application to Risk Measurement for NSE 20 Share Index

We compare risk estimates from VaR, ES and WES using NSE[3] 20 Share index data from January 2008 to March 2021. Returns and loss returns are calculated from the price data using Equations (1) and (2) respectively.

Table 1 reports summary statistics of the data which shows that the data is heavy tailed and skewed to the left. Moreso, the reported p-value from the ADF test implies that the data is stationary at 5% significance level.

From **Figure 1** we observe some level of volatility clustering which is common in most financial data sets.

[3]Nairobi Stock Exchange.

Table 1. Sample statistics.

Statistical properties of the loss return data	
minimum	−0.09018
maximum	0.08243
median (Q_1, Q_3)	0.00013 (−0.00374, 0.00406)
mean ± sd	0.00017 ± 0.00843
skewness	−0.37124
kurtosis	16.04896
ADF test p-value	0.01

Q_1, Lower quartile Q_3, Upper quartile sd, standard deviation.

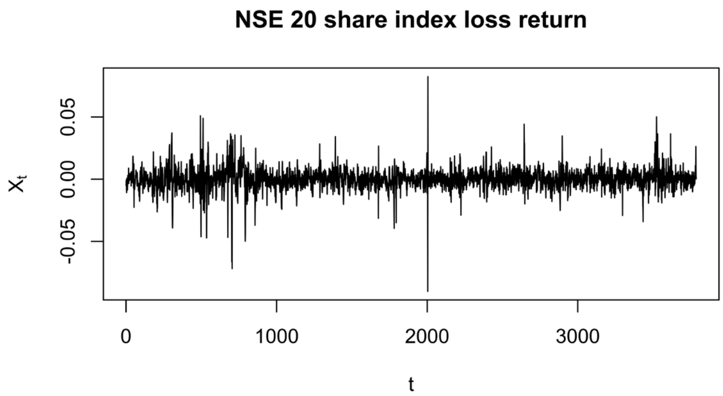

Figure 1. NSE 20 share, loss return time series.

The ACF and PACF plots of the data in Figure 2 suggests autocorrelation in the series of up to order two which informs the number of lags in our QAR process. Therefore using extreme quantile autoregression we obtain the following model for the loss return

$$X_t = -0.00008188 + 0.2932X_{t-1} + 0.1330X_{t-2} + \left(0.002647 + 0.1749\left|\varepsilon_{t-1}\right| + 0.06753\left|\varepsilon_{t-2}\right|\right)Z_t \quad (34)$$

where Z_t follows the extreme value distribution given by Equation (11).

Based on p-values in Table 2 all the the estimated parameters of the quantile process are significant at 5% significance level except the constant of the central quantile process.

The ACF plot in Figure 3 confirms independence of the resulting standardized errors. Hence using the threshold $u = 2.038538$, to ensure that 10% of the errors are classified as extreme, we obtain the following estimates of the shape and scale parameters from the GPD fit.

As outlined in Table 3, the p-values of all the parameter estimates are less than 5% significance level implying that the distribution fits the data well. Figure 4 shows a plot of the corresponding risk estimates at 97.5% confidence level.

From Figure 4 we observe that risk estimates from ES are slightly higher than those from WES confirming that WES is less conservative compared to

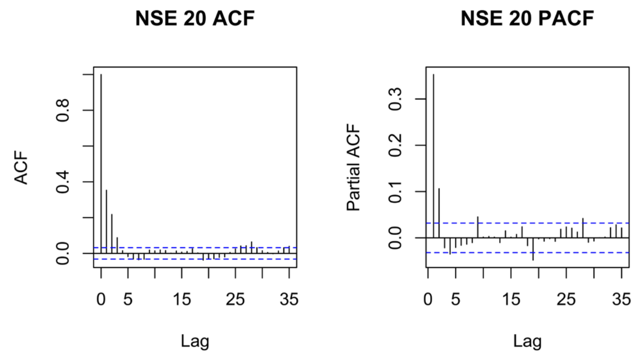

Figure 2. Autocorrelation and Partial Autocorrelation Plot of the loss return series.

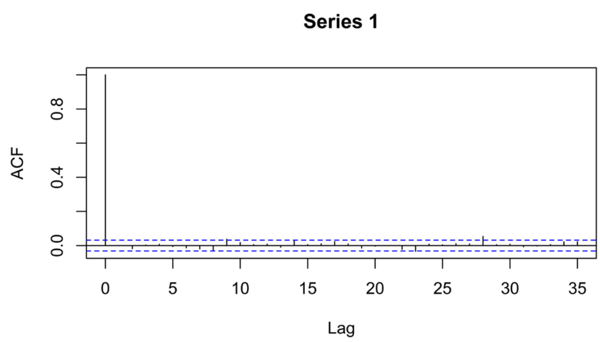

Figure 3. Autocorrelation Plot of the standardized errors.

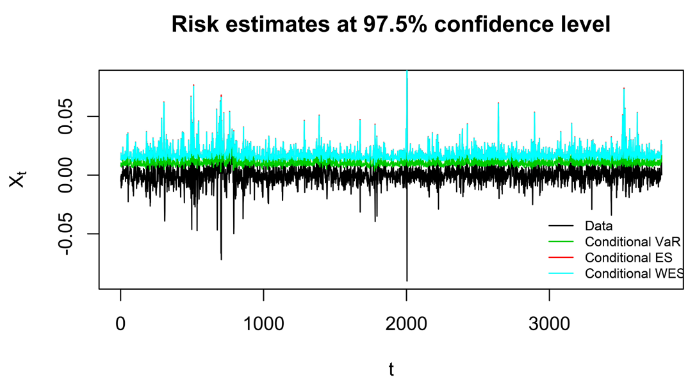

Figure 4. Data plot superimposed with risk estimates from considered Risk measures.

Table 2. Parameter estimates of the central quantile and the scale.

	Term	Estimate	Std. Error	t-value	p-value
	constant	0.00008187868	0.0001184322	0.6913549	0.24471
Central	QAR(1)	0.2931358	0.0251979805	11.6333058	0.0000
	QAR(2)	0.1329694	0.0227462397	5.8457766	0.0000
	constant	0.002646745	0.0001448444	18.273030	0.0000
Scale	QAR(1)	0.174948799	0.0273538824	6.395758	0.0000
	QAR(2)	0.067533867	0.0231822250	2.913175	0.0018

Table 3. Parameter estimates of the Central quantile and the scale.

Parameter	Estimate	Std. Error	t-value	p-value
Shape	0.2445236	0.06464320	3.7826546	0.0000
Scale	0.9666750	0.07864687	12.2913347	0.0000

ES. However, risk estimates from both ES and WES are higher than those from VaR because VaR failed to appreciate severity of losses beyond the 97.5% confidence threshold.

6. Conclusion and Recommendations

We have improved the adjusted extreme conditional quantile estimator in [11] and used it to obtain the one-step-ahead risk estimators for Var, ES and WES. The three estimators were then used to quantify risk in the NSE 20 share index portfolio. Consistency and coherence of the proposed Weighted expected shortfall were also proved. It was observed that the WES is less conservative compared to ES.

Data Availability

The authors used NSE 20 share index data obtained from the Nairobi Stock Exchange (NSE).

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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