



MACHAKOS UNIVERSITY

UNIVERSITY EXAMINATIONS 2018/2019

**FIRST YEAR SECOND SEMESTER EXAMINATIONS FOR THE DEGREE OF
BACHELOR OF SCIENCE IN ~~ACTUARIAL SCIENCE~~**

SAC 100: ~~PRINCIPLES OF ACTUARIAL SCIENCE~~

DATE : _____ TIME: 2 HOURS

INSTRUCTIONS: Attempt Question ONE and any other TWO questions.

QUESTION ONE (30 MARKS)

- a. Define the following terms:
- i. A differential Equation. (1 mark)
 - ii. Half-life. (1 mark)
 - iii. Doubling time. (1 mark)
 - iv. Equilibrium state. (1 mark)
 - v. Interactions. (1 mark)
- b. Explain two types of growth rates. (2 marks)
- c. Find both equilibrium solutions of $\frac{ds}{dt} = 2s^2 + s - 15$ and determine their stability. (4 marks)

- d. Find the particular solution to $\frac{dX}{dt} = \frac{3X+5}{2}$ satisfying $X(0) = 7$. (4 marks)
- e. Verify that a fixed point of $Y_{n+1} = \frac{3Y_n^2 - 6Y_n + 8}{4}$ is $Y_{eq} = 2$ and find the other fixed point. Determine the stability of both fixed points. (6 marks)
- f. List three applications of Logistic Models. (3 marks)
- g. Solve $Q_{n+2} = 5Q_{n+1} - 6Q_n$ with $Q_0 = 0, Q_1 = 2$. (6 marks)

QUESTION TWO (20 MARKS)

1. The interaction between a host species and a parasite species is modelled by the following pair of differential equations

$$P'(t) = 7P(t) - 4Q(t)$$

$$Q'(t) = 5P(t) - 2Q(t)$$

Where $P(t)$ and $Q(t)$ give the populations of the host and parasites at time t respectively. The time is measured in years.

- i. Find the general solutions for $P(t)$ and $Q(t)$. (4 marks)
 - ii. Find the particular solution for $P(0) = 1500$ and $Q(0) = 2000$ and use it to determine the long term behavior of both species. (4 marks)
 - iii. Assume that $P(0)$ and $Q(0)$ are both positive and let R represent the average number of parasites per host when $t = 0$. For which range of values of R do both species survive indefinitely. (3 marks)
2. Determine the equilibrium and stability of $\frac{dp}{dt} = \cos Q$. (2 marks)

Hence:

- i. Sketch the growth rate curve (Phase diagram), (4 marks)
- ii. Sketch the general solution. (3 marks)

QUESTION THREE (20 MARKS)

1. Consider the prey predator system below, where x represents the prey and y the predators.

i.
$$\begin{aligned}\frac{dx}{dt} &= 3x - 9xy \\ \frac{dy}{dt} &= -y + 4xy\end{aligned}$$

ii.
$$\begin{aligned}\frac{dx}{dt} &= 7x - \frac{1}{3}xy \\ \frac{dy}{dt} &= -y + \frac{1}{4}xy\end{aligned}$$

- a. In which system does the prey reproduce more quickly when there are no predators (justify your answer)? (2 marks)
- b. In which system are the predators more successful at catching prey (justify your answer)? (2 marks)
- c. Modify the first model (i.) in such a way that it includes the effect of hunting of predator at a rate $\alpha = 0.2$ proportional to the number of predator. (1 mark)
- d. Suppose in the second model (ii.) that in the absence of predator, the prey population grows logistically with a carrying capacity of 80. Write the system which takes into account the above assumption. (2 marks)
2. A public health campaign causes the contagiousness of a disease to decay exponentially. The spread of the epidemic can now be modelled by the equation $\frac{dp}{dt} = re^{-at}p(1-p)$ where p represents the fraction of the population with the disease, r and a are both positive constants and t is the time in days.
- a. Use separation of variables and partial fractions to determine the general solution to the above differential equation. (5 marks)
- b. If $r = 0.2$, $a = 0.04$ and $p(0) = 0.1$ calculate the time t when half of the population is infected. (3 marks)
- c. Using the same parameters given above, calculate the fraction of the population that is infected when $t = 40$. (3 marks)

- d. Let $R(t) = \frac{p(t)}{1-p(t)}$ be the ratio of infected to uninfected individuals. Use the general solution to show that $\lim_{t \rightarrow \infty} R(t) = R(0)e^{r/a}$. (2 marks)

QUESTION FOUR (20 MARKS)

1. Consider the one parameter family model described by the equation

$$\frac{dy}{dt} = y^3 + \alpha y + y$$

- a. Locate the bifurcation value and describe the bifurcation that takes place. (4 marks)

- b. Draw the bifurcation diagram. (3 marks)

2. Consider a population described by the differential equation $\frac{dy}{dt} = y^2 - 4y + 2$.

- a. Find the equilibrium points, their stability and draw the phase diagram. (4 marks)

- b. Describe the long-term behavior of the population with the given initial population:

i. $y(0) = 2$ (2 marks)

ii. $y(0) = 5$ (2 marks)

3. Find the particular solution to $\frac{dx}{dt} = 0.1x(4-x)$, where $x(0) = 5$. (5 marks)

QUESTION FIVE (20 MARKS)

1. The interaction of two biochemical reagents is monitored at regular intervals and can be modelled by the following pair of coupled recurrence relations

$$X_{n+1} = \frac{6}{5}X_n + \frac{7}{10}Y_n$$

$$Y_{n+1} = \frac{3}{10}X_n + \frac{4}{5}Y_n$$

Where X_n and Y_n denote the amount of each reagent after n observations.

- a. Show that the characteristic equation is $r^2 - 2r + \frac{3}{4} = 0$. (2 marks)
 - b. Find the general solution for X_n and Y_n . (4 marks)
 - c. Find the particular solution when $X_0 = 900$ and $Y_0 = 100$. (3 marks)
 - d. Show that as $n \rightarrow \infty$ the ratio X_n/Y_n tends to $7/3$. (1 mark)
 - e. Show that the limit in the answer to part (d) does not depend on the initial amounts. (2 marks)
2. Given $X_{n+1} = \frac{1}{6}X_n^2(5 - X_n)$. Find:
- a. Fixed points. (2 marks)
 - b. Equilibrium condition and stability. (3 marks)
 - c. Approximate general solution. (3 marks)