MACHAKOS UNIVERSITY

University Examinations 2016/2017

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR FIRST SEMESTER EXAMINATION FOR DEGREE IN BACHELOR OF EDUCATION (SCIENCE)

SMA 403: GALOIS THEORY

DATE: 30/5/2017 **TIME: 8:30 – 10:30 AM**

INSTRUCTIONS

Answer question ONE (Compulsory) and any other TWO questions

QUESTION ONE: 30 MARKS (COMPULSORY)

a) Give brief definitions of each of the following and in each case, give a simple example;

i) A field (2 marks)

ii) An automorphism of a field. (2 marks)

iii) A Galois Group (2 marks)

b) i) Let E be an algebraic extension of a field F. Give the definition of the concept of conjugate elements of E over F. (2 marks)

ii) Find the zeros of the polynomial $f(x) = x^4 - 5x^2 + 6$ over. Deduce the splitting field of this polynomial. (4 marks)

iii) Using (i) and (ii) construct the Galois group $(E\backslash Q)$ (6 marks)

iv) State the fundamental theorem of Galois theory then do a demonstration by constructing lattice diagrams of the subgroups of the Galois group and the subfields of the splitting field E above. (6 marks)

c) Let $\{\sigma_i : i\epsilon I\}$ be a collection of automorphism of a field E, show that

$E(\sigma_i) = \{a \in E: \sigma_i(a) = a \text{ for all } \sigma_i , i \in I\}$ is a subfield of E. (6 marks)

QUESTION TWO (20 MARKS)

a) Let F be afield. Define the ***n^{th}cyclotomic polynomial*** over the field F . (2 marks)

b) (i) Compute a primitive eighth root of unity in $\boldsymbol{C}$. i.e. a root of the polynomial:

$$f(x) = x^8 - 1. \qquad \text{(3 marks)}$$

(ii) Write down in terms of this primitive root all the other primitive roots. (3 marks)

(iii) Write down the ***8^{th}cyclotomic polynomial*** $\Phi_8(x)$. (2 marks)

c) Let $K = Q(\alpha)$ be the splitting field of $f(x) = x^8 - 1$, where α is a primitive root in (b)(i) above. Show that the Galois group $G(K\backslash Q)$ is isomorphic to the group G_n of all positive integers relatively prime to n under multiplication modulo n. Conclude that this group has order $\varphi(n)$, the Euler phi-function that counts this integers.
(Hint: Define an automorphism $\alpha_m\colon G(K\backslash Q) \to G(K\backslash Q)$, $\alpha \to \alpha^m$, where α , α^m are conjugates over Q, and m is a positive integer) . Prepare the cayley multiplication table for this group. (7 marks)

d) Prove that the Galois group of the p th cylotomic extension of Q, for a prime p is cyclic of order $p - 1$. (3 marks)

QUESTION THREE (20 MARKS)

a) Consider the Field E = $Q(\sqrt{2}, \sqrt{3}, \sqrt{5})$
Show that $[E: Q] = 8$ (4 marks)

b) Let $\tau_2 = \gamma_{\sqrt{2},\ -\sqrt{2}}$, $\tau_3 = \gamma_{\sqrt{3},\ -\sqrt{3}}$, $\tau_5 = \gamma_{\sqrt{5},\ -\sqrt{5}}$
Compute the indicated elements of E.

i) $\tau_2(\sqrt{2} + \sqrt{5})$,

ii) $(\tau_3\tau_2)(\sqrt{2} + 5\sqrt{3})$,

iii) $(\tau_5\tau_3)\left(\frac{\sqrt{2}-3\sqrt{5}}{2\sqrt{3}-\sqrt{2}}\right)$ (6 marks)

c) Using (a) i find the fixed field of the automorphism(s) of E for;
(i)$(\tau_5\tau_2)$ (ii) $(\tau_5\tau_3\tau_2)$ (iii) $(\tau_5{\tau_3}^2)$ (6 marks)

d) Find the subgroup H of $G(E\backslash Q)$ generated by the elements τ_2 , τ_3 , τ_5 and construct the group table. (Hint: There are 8 elements.) (4 marks)

QUESTION FOUR (20 MARKS)

a) Briefly explain what is meant by an extension $K$of a field F by radicals. (2 marks)

b) Let $E = Q\left(\sqrt{3}, \sqrt{7}\right)$ be an extension of the field Q by radicals.

(i) Describe and identify the elements of the Galois group $G(Q\left(\sqrt{3}, \sqrt{7}\right)/Q)$ (4 marks)

(ii) Demonstrate that $G(Q\left(\sqrt{3}, \sqrt{7}\right)/Q)$ is isomorphic to a subgroup of the symmetric group S_4 . (You may use the numbers 1,2,3,4) (4 marks)

(iii) Let $u = \sqrt{3} + \sqrt{7}$ be a primitive root of the minimal polynomial in $Q\left(\sqrt{3}, \sqrt{7}\right)$ over Q. Using (i) above determine this minimal polynomial. (4 marks)

c) Find a primitive element in the splitting field in each of the following polynomials over $Q[x]$.

(i) $x^4 - 8x^2 + 15$

(ii) $x^3 - 2$ (6 marks)

QUESTION FIVE (20 MARKS)

a) Find the splitting field K in C of the polynomial $x^4 - 4x^2 - 1$ over Q. Compute the group of the polynomial over Q and exhibit the correspondence between the subgroup of $G(K\backslash Q)$ and the intermediate fields. (Draw diagram) (6 marks)

b) Derive the cubic formula for finding the zeros of the polynomial

$$f(x) = ax^3 + bx^2 + cx + d.$$ (6 marks)

c) Give the definition of a symmetric function. (1 marks)

d) Determine the solvability by radicals of the polynomial $f(x) = x^5 - 7.$ (2 marks)

e) Briefly discuss the insolvability of the general quintic using (c) above. (5 marks)