MACHAKOS UNIVERSITY

University Examinations for 2021/2022

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FIRST YEAR SECOND SEMESTER EXAMINATION FOR

MASTER OF SCIENCE (MATHEMATICAL MODELING AND COMPUTATION)

SMA 808: FINITE DIFFERENCE AND FINITE VOLUME METHODS

DATE: 24/1/2022 **TIME: 10:00 – 1:00 PM**

INSTRUCTIONS: Answer any THREE questions

QUESTION ONE (20 MARKS)

a) The following MATLAB code implements the finite difference method to solve a given Poisson equation subject to the specified boundary conditions;

```
function FDPoissonEqn()
clear all;clc;
x0=0; xn=2; n=16; dx=(xn-x0)/n; x=x0:dx:xn;
y0=0; ym=1; m=15; dy = (ym-y0)/m; y=y0:dy:ym;
%%======================
A=(dy*dy)/(2*(dx^2+dy^2));B=(dx*dx)/(2*(dx^2+dy^2));
C=((dx*dy)^2)/(2*(dx^2+dy^2));
%%======================
u=zeros(n,m);
%%boundary conditions specification
u(1,:)=0; u(n,1:m)=2*exp(y(1:m));
u(1:n,1)= x(1:n);u(1:n,m)=exp(1)*x(1:n);
%% approximate solution by finite difference method
  for i=2:n-1
    for j=2:m-1
   u(i,j)=A*(u(i+1,j)+u(i-1,j))+B*(u(i,j+1)+u(i,j-1))-
C*x(i)*exp(y(j));
    end
  end
mesh(u(:,:,K))% this is for approximation solution
end
```

i) Modify the code to implement the Gauss-Siedel iteration method in order to improve the approximation of the solution. (7 marks)

ii) Given the exact solution to the problem as $u(x, y) = xe^y$, write the code segments needed to compute the exact values within `for` loops and plot the curves of both the exact and approximate solution at $y = 0.5$, on the same graph. (5 marks)

b) A domain $[0, L] \times [0, T]$ in the $x - t$ plane is discretized such that the extreme values $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$ of cell i define the position of intercell boundaries;

i) Define the average value of a function $u(x, t)$ in cell i at a fixed time $t = n\Delta t$. (2 marks)

ii) For a conservation law $\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0$ describe the three (3) stages of the Godunov numerical method that leads to determination of new piecewise constant data $w_i^{n+1}(x, t^{n+1})$ (6 marks)

QUESTION TWO (20 MARKS)

a) Briefly describe the following concepts of numerical schemes for conservation laws;

i) Total Variation Diminishing (TVD) (2 marks)

ii) Slope limiters (2 marks)

b) The air pressure $p(x, t)$ in an organ pipe is governed by the wave equation $\frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 p}{\partial t^2}, \quad 0 < x < L, \ t > 0$ where L=1 is the length of the pipe. Let the pipe be open such that the boundary conditions are given as $p(0, t) = 0.9$, $p(1, t) = 0.9$ and the initial conditions are given by $p(x, 0) = 0.9 \cos 2\pi x$, $\frac{\partial p(x,0)}{\partial t} = 0, \quad 0 \leq x \leq 1$.

i) Determine the finite difference equation whereby the approximate variable $w(i, n + 1)$ is the subject of the formula. (4 marks)

ii) Write a MATLAB code that implements finite difference method to approximate the pressure and visualizes the solution as a surface plot. Take $\Delta x = \Delta t = 0.1$. (12 marks)

QUESTION THREE (20 MARKS)

a) Suppose, during rush hours, there is a one-way road section (without exit and entry on the sides) located between $x = -10$ and $x = 10$ with a traffic light located at $x = 0$, and vehicles are moving towards $x = 10$

i) If the traffic light turns red, for time $t = T$, then sketch a representation of shock wave formation in the flow of traffic on this road section. (3 marks)

ii) If the traffic light turns green, for time t=T, then sketch a representation of rarefaction fan formation in the flow of traffic on this road section. (3 marks)

iii) Briefly explain how to determine traffic density and flow rate on the road. (4 marks)

b) Given a domain $R = \{(x, y) | 0 < x < 2, 0 < y < 2\}$;

i) Discretize it using the step size $\Delta x = \Delta y = 0.5$ and sketch mesh grid. (4 marks)

ii) Write a MATLAB code that appropriately discretizes R and prescribes the following boundary conditions on the four sides of R for some function $u(x, y)$;

$$u(x, 0) = x^2, \quad u(x, 2) = (x - 2)^2, \; 0 \le x \le 2$$
$$u(0, y) = y^2, \quad u(2, y) = (y - 1)^2, \; 0 \le y \le 2$$

(6 marks)

QUESTION FOUR (20 MARKS)

a) Consider the heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \; 0 < x < L, \; 0 < t < T$ together with the boundary conditions $u(0, t) = u(L, t) = 0, 0 < t < T$, and initial condition $u(x, 0) = f(x), \; 0 \le x \le L$.

i) Determine the forward difference approximation method at time t_j, and the backward difference approximation method at time t_{j+1} of the heat equation. (3 marks)

ii) Obtain the Crank-Nicolson approximation method for the heat equation and represent it in the matrix form $Aw^{(j+1)} = Bw^{(j)}$ for each $j = 0,1,2, \ldots$ where A and B are tridiagonal matrices. (7 marks)

iii) The MATLAB code below implements the above matrix form to solve a given heat equation. Add the missing code segments indicated in lines 14 to 23 for the specification of matrices. (10 marks)

```
1. function CNheatEqn()
2.   clear all; clc;
3.   x0=0;xm=1;m=10;h=(xm-x0)/m
4.   t0=0;tN=0.5;N=50;k=(tN-t0)/N
5.   x=x0:h:xm; t=t0:k:tN;
6.   alpha=1;lambda=(k*alpha^2)/(2*h*h);
7.   u=zeros(m,N);
8.   %% boundary and initial conditions
9.      u(1,:)=0;u(m,:)=0;
10.    for i=1:m
11.       u(i,1)=sin(pi*x(i));
12.    end
13.   %% specification of tridiagonal matrices A and B
14.   ...............
15.   ...............
16.   ...............
17.   ...............
18.   ...............
19.   ...............
20.   %% solve the matrix form of the problem
21.   ..................
```

```
..................
..................
%% visualization of the approximate solution
 figure(1)
   mesh(u)
   title('CN method')
end
```

QUESTION FIVE (20 MARKS)

a) Aided by an appropriate diagram, illustrate the concept of Courant-Friedrich-Lewy condition on timestep Δt for conservation laws. (4 marks)

b) Consider the Euler equations $\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0$ whereby $U = [\rho, \rho v]^T,\ F(U) = [\rho v,\ \rho v^2 + \rho a^2\ ]^T$;

i) Reduce the Euler equations into $\frac{\partial \bar{u}}{\partial t} + A(\bar{u})\frac{\partial \bar{u}}{\partial x} = 0$ whereby $\bar{u} = [\rho,\ v]^T$ and obtain the wave speeds $\lambda_1 = v - a,\ \lambda_2 = v + a.$ (7 marks)

ii) Apply the Rankine-Hugoniot jump condition to determine the expressions of the intermediate state variable Q^* and shock speed S in terms of variable ρ^*. Note that $\rho v = Q$. (9 marks)