MACHAKOS UNIVERSITY

University Examinations for 2020/2021 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR FIRST SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS)

SMA 409: ALGEBRAIC GEOMETRY

DATE: 12/8/2021 **TIME: 8:30 - 10:30 AM**

INSTRUCTIONS:

Answer question ONE and any other TWO questions

QUESTION ONE (COMPULSORY) (30 MARKS)

Describe what is meant by Algebraic variety. Let $S = \{3x - 8y - 24\} \subset \mathbb{R}[x, y]$ describe the algebraic variety $V(S)$ in $\mathbb{R}^2$. (4 marks)

a) Generate the set of all polynomials of degree ≤ 3 in $\mathbb{Z}_2[x]$. (3 marks)

b) Determine the zeros of the equation $x^2 + 2x + 4 = 0$ in $\mathbb{Z}_6$. (4 marks)

c) Show that the set of common zeros in $F^n$of the polynomials $f_1, f_{2,\ldots}f_n \in F[x]$,where F is a ring is the same as set of common zeros in$F^n$of all the polynomials in the entire ideal $I = \langle f_1, f_2, \ldots, f_n \rangle \in F[x]$ (5 marks)

d) Briefly describe each of the following terms as used in algebraic geometry

i. Fundamental theorem of algebra. (2 marks)

ii. Nullstellensatz for $\mathbb{C}[x]$ (3 marks)

e) Given the evaluation homomorphism $\phi_\alpha: F[x] \to \mathbb{C}$, compute $\phi_{-3,2i}(x^2y^3 + 3x^4y)$ (4 marks)

f) Let F be a field. State the uniqueness factorization theorem in $F(x)$ hence factorize $f(x) = x^4 + 4$ into linear factors in $Z_5(x)$ (5 marks)

QUESTION TWO (20 MARKS)

a) Distinguish between Affine n -space (A^n) and a Projective space (P^n). (4 marks)

b) Describe what is meant by The Zariski Topology on $A^n[F]$. (2 marks)

c) Prove that the Zariski Topology on $A^n[F]$ is indeed a topological space. (4 marks)

d) Find a Grobner basis for the ideal $\langle x^2y - x - 2,\ xy + 2y - 9\rangle$ and show that the algebraic variety is $\left\{(1,3), (-\frac{1}{2}, 6)\right\}$. (6 marks)

e) By division, reduce the basis $\{\ xy^2,\ y^2 - y\}$ for an ideal $I =< xy^2,\ y^2 - y >$ in $R[x, y]$ to one with smaller term size assuming the order lex with $y < x$. (4 marks)

QUESTION THREE (20 MARKS)

a) Describe what is meant by an evaluation homomorphism. (2 marks)

b) Compute the following evaluation homomorphisms and comment:

i) $\emptyset_i(2x^3 - x^2 + 3x + 2)$ (3 marks)

ii) $\emptyset_5(x^3 + 2)(4x^2 + 3)$ (3 marks)

c) Let $S = \{2x - 6y - 14\} \subset \mathbb{R}[x, y]$ describe the algebraic variety $V(S)$ in $\mathbb{R}^2$. (4 marks)

d) State the Hilbert basis theorem. Use an example to illustrate. (3 marks)

e) Describe the algebraic variety $V(< x + y - 3z - 8, 2x + y + z + 5 >)$ in R^3 . (5 marks)

QUESTION FOUR (20 MARKS)

a) Identify any two examples of Affine algebraic sets. (2 marks)

b) Show that $\{\ x + y,\ y^2 - y + 1\}$ is a grobner basis for $< x^2y + x + 1,\ xy^2 + y - 1 >$, hence verify that the variety consists of one point $(a, -a)$, where $a \simeq 1.3247$. (6 marks)

c) Describe the algebraic variety $V(< f(x), g(x) >)$ in R^3 where, $f(x) = x^4 + x^3 - 3x^2 - 5x - 2$, $g(x) = x^3 - 3x^2 - 6x - 8$ (6 marks)

d) State and prove strong Hilbert Nullstellensatz. (6 marks)

QUESTION FIVE (20 MARKS)

a) Let F be a field. Show that if S is nonempty subset of F^n then $I(S) = \{f(x) \in F[x]: f(S) = 0 \ \forall \ s \in S$ is an ideal of $F[x]$ (6 marks)

b) Briefly describe each of the following giving examples.

i. Algebraic element. (4 marks)

ii. Transcendental element. (4 marks)

c) Show that if $f(x) = g(x)q(x) + r(x)$ then the common divisorsin $F[x]$ of $f(x)$ and then the common divisors in $F[x]$ of $f(x)$ are the same as the common divisors in the $F[x]$ of $f(x)$ and $r(x)$. (6 marks)