MACHAKOS UNIVERSITY

University Examinations for 2022/2023 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR ……. SEMESTER EXAMINATIONS FOR

BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)

BACHELOR OF SCIENCE (STATISTICS AND PROGRAMMING)

BACHELOR OF SCIENCE (MATHEMATICS)

BACHELOR OF EDUCATION (SCIENCE)

SMA 303: RING THEORY

DATE: **TIME:**

INSTRUCTION: *Answer Question **ONE** which is compulsory and any other **TWO** Questions*

QUESTION ONE (COMPULSORY) (30 MARKS)

a) Explain the meaning of the following terms:

(i) an *integral domain*, (3 marks)

(ii) a *right ideal* in a ring R. (3 marks)

b) Suppose that R is a ring and let $a, b, c, d \in R$ be elements of R.

(i) State the distibutivity axioms in a ring R. (4 marks)

(ii) Prove that $(a + b)(c + d) = ac + ad + bc + bd$. (4 marks)

c) Let R be a ring and consider the subset $Z(R) = \{a \in R: ay = ya \text{ for all } y \in R\}$ i.e. $Z(R)$ is the center of R. Prove that $Z(R)$ is a subring of R. (4 marks)

d) Let $(\mathbb{Z}_{16}, \oplus, \otimes)$ denote the ring of integers modulo 16 under addition and multiplication modulo 16. Let I be the *principal ideal* of $\mathbb{Z}_{16}$ generated by 4.

(i) Determine all elements of I. (4 marks)

(ii) Construct all the elements of the *quotient ring* $\mathbb{Z}_{16}/I$. (4 marks)

e) Let A be a ring with identity $f: A \to B$ be a surjective ring homomorphism. Prove that B is also a ring with identity. (4 marks)

QUESTION TWO (20 MARKS)

a) Distinguish between a *zero divisor* and a *unit* in an ring R. (4 marks)

b) Let R be a ring such that $(a+b)^2 = a^2 + b^2 + 2ab$ for any $a, b \in R$. Show that R is a commutative ring. (5 marks)

c) Let $\mathbb{Z}[i] = \{x + iy \mid x, y \in \mathbb{Z}\} \subset \mathbb{C}$ be the set of Gaussian integers. Determine all the units in $\mathbb{Z}[i]$. (5 marks)

d) Suppose that $\varphi: A \to B$ is a surjective ring homomorphism. Prove that

(i) if $I \subset A$ is a prime ideal then the image $\varphi(I)$ is a prime ideal of B, (3 marks)

(ii) if $J \subset B$ is a prime ideal then $\varphi^{-1}(J) = \{a \in A \mid \varphi(a) \in J\}$ is also a prime ideal.

(3 marks)

QUESTION THREE (20 MARKS)

a) Explain the meaning of the following terms.

(i) a *subring* S in a ring R, (2 marks)

(ii) a *principal ideal* in a ring R. (2 marks)

b) Let $M_2(\mathbb{C})$ be the ring of all 2×2 matrices with complex entries. Consider the subset $T \subset M_2(\mathbb{C})$ given by

$$D = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{C}\right\}$$

i.e. T is the subset containing upper triangular matrices. Prove that T is a subring of $M_2(\mathbb{C})$.

(5 marks)

c) Suppose that A and B are rings. Prove that if $\varphi: A \to B$ is a ring homomorphism then

(i) $\varphi(A)$ is a subring of B, (4 marks)

(ii) $Ker\ \varphi$ is an ideal of A, (4 marks)

(iii) if A is a commutative ring then $\varphi(A)$ is commutative. (3 marks)

QUESTION FOUR (20 MARKS)

a) Explain the meaning of the following terms (4 marks)

(i) a *prime ideal* in a ring R,

(ii) a *maximal ideal* in a ring R.

b) Suppose that U and V are ideals of a ring R. Prove that $U \cap V$ is also an ideal of R.

(4 marks)

c) Let $S = \{m + n\sqrt[3]{2} : m, n \in \mathbb{Z}\} \subset \mathbb{R}$. Determine whether S is a subring of $\mathbb{R}$. (4 marks)

d) Let R be a ring and $f: \mathbb{Q} \to R$, $g: \mathbb{Q} \to R$ be ring homomorphisms such that $f(n) = g(n)$ for every $n \in \mathbb{Z}$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Hint: Every element of $\mathbb{Q}$ can be written as $\frac{m}{n}$ for some $m, n \in \mathbb{Z}$ with $n \neq 0$. (8 marks)

QUESTION FIVE (20 MARKS)

a) Consider the binary operations of addition and multiplication defined respectively on $\mathbb{R}^2$ by:

$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and $(x_1, x_2) \cdot (y_1, y_2) = (x_1y_1 - x_2y_2, x_1y_2 + x_2y_1)$.

(i) Prove that $(\mathbb{R}^2, +, \cdot)$ is a ring. (6 marks)

(ii) Consider the map $\psi: \mathbb{R}^2 \to \mathbb{R}$ defined by $\psi(x, y) = x + y$. Determine if ψ a ring homomorphism. (4 marks)

b) Let $\mathbb{Q}[\sqrt{3}]$ denote the set $\{x + y\sqrt{3} : x, y \in \mathbb{Q}\}$. Define the binary operations $\oplus$ and $\otimes$ on $\mathbb{Q}[\sqrt{3}]$ as follows

$$\begin{aligned}(x + y\sqrt{3}) \oplus (z + w\sqrt{3}) &= (x + y) + (y + w)\sqrt{3} \\ (x + y\sqrt{3}) \otimes (z + w\sqrt{3}) &= (xz + 3yw) + (xw + yz)\sqrt{3}\end{aligned}$$

(i) Prove that $\mathbb{Q}[\sqrt{3}]$ is a subring of $\mathbb{R}$. (4 marks)

(ii) Determine if every non-zero elements of $\mathbb{Q}[\sqrt{3}]$ is invertible. (6 marks)