MACHAKOS UNIVERSITY

University Examinations for 2020/2021

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

THIRD YEAR APRIL SESSION EXMINATION FOR

BACHELOR OF EDUCATION (SCIENCE)

BACHELOR OF EDUCATION (ARTS)

SMA 336: ORDINARY DIFFERENTIAL EQUATION II

DATE: 19/7/2021 **TIME: 11:00 – 1:00 PM**

INSTRUCTION: *Answer Question* ***ONE*** *which is compulsory and any other* ***TWO*** *Questions*

QUESTION ONE (COMPULSORY) (30 MARKS)

a) State the existence and uniqueness theorem for the nth order linear differential equation. (2 marks)

b) i) Define the Wronskian of a set of functions $F = \{f_1\ ,\ f_2\ ,\ f_3\ \ , \ldots ,\ f_n\ \}$(3 marks)

ii) Show that the Wronskian of the functions x^2 *and* $x^2 \ln x$ is non-zero. (4 marks)

c) i) Consider the nth order differential equation

$a_0 \dfrac{d^n y}{dx^n} + a_1 \dfrac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1} \dfrac{dy}{dx} + a_n y = F$. State three conditions that the equation could be an nth order linear differential equation. (3 marks)

d) ii) Determine a first integral of the differential equation $\dfrac{dy}{dx}\dfrac{d^2 y}{dx^2} - x^2 y \dfrac{dy}{dx} = xy^2$ (3 marks)

e) Solve the equation $y'' - y = x$ at $x = 0$ using the power series solution method. (3 marks)

f) Consider the second order linear differential equation $(2x+1)\dfrac{d^2 y}{dx^2} - 4(x+1)\dfrac{dy}{dx} + 4y = 0$. Determine the linearly independent solution of the equation by reducing the order. (6 marks)

g) Consider the differential equation $3x^2dx + 3y^2dy - \left(x^3 + y^3 + e^{2z}\right)dz = 0$; solve the equation by taking one of the variables as a constant. (6 marks)

QUESTION TWO (20 MARKS)

a) Define linearly independent solution of a linear system of differential equations.(3 marks)

b) consider the system of differential equations

$$\frac{dx}{dt} = 2x - y$$
$$\frac{dy}{dt} = 3x + 6y$$

i.) Show that $\begin{matrix} x = e^{5t} \\ y = -3e^{5t} \end{matrix}$ and $\begin{matrix} x = e^{3t} \\ y = -e^{3t} \end{matrix}$ are linearly independent solutions of the system (6 marks)

ii.) Determine the solution $x = f(t)$, $y = g(t)$ of the system which is such that $f(0) = 0$, $g(0) = 2$ (4 marks)

c) Use the operator method to determine the general solution of the linear systems.

$$(D-1)x + Dy = 2t + 1$$
$$(2D+1)x + 2Dy = t$$

(7 marks)

QUESTION THREE (20 MARKS)

Consider the system of differential equations below;

$$\frac{dx}{dt} = -2x + 2x^2$$
$$\frac{dy}{dt} = -3x + y + 3x^2$$

a) Determine;

i.) the critical points (4 marks)

ii.) the nature of the critical points (3 marks)

b) Linearize the system of the differential equation (8 marks)

c) Solve the system for x *and* y. (5 marks)

QUESTION FOUR (20 MARKS)

a) Consider the differential equation

$$x^2 \sin x \frac{d^3 y}{dx^3} - \left(3x^2 \sin x + x^3 \cos x\right)\frac{d^2 y}{dx^2} + \left(6x \sin x + 2x^2 \cos x\right)\frac{dy}{dx} - \left(6 \sin x + 2x \cos x\right)y = 0$$

i) Show that $y = x$ is a solution of the differential equation above. (3 marks)

ii) Reduce the third order equation to a second order equation and solve it. (7 marks)

b) Consider the following differential equation;

$$\left(z + z^3\right)\cos x dx - \left(z + z^3\right)dy + \left(1 + z^2\right)\left(y - \sin x\right)dz = 0$$

i) Verify the integrability condition for the equation (3 marks)

ii) Solve the equation (7 marks)

QUESTION FIVE (20 MARKS)

a) Consider the Legendre equation $\left(1 - x^2\right)y'' - 2xy' + p\left(p + 1\right)y = 0$. This equation has two independent series solution ;

$$y = c_0\left[1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p-2)(p+1)(p+3)}{4!}x^4 + \ldots\right]$$
$$+ c_1\left[x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!}x^5 + \ldots\right]$$

Which is denoted by $y = c_0 y_1 + c_1 y_2$. Show that the Legendre polynomial of order three is given by $p_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$ (3 marks)

b) The Rodriquez formula for Legendre's polynomial of order n is given by;

$$p_n(x) = \frac{1}{2^n n!}\frac{d}{dx^n}\left(x^2 - 1\right)^n$$

Determine the Legendre's polynomial for;

i.) $p_1(x)$ (2 marks)

ii.) $p_2(x)$ (3 marks)

c) Prove that;

i.) $J_{-p}(x) = (-1)^p J_p(x)$ (4 marks)

ii.) Bessel's function of order zero is given by

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots$$ (4 marks)

d) Show that the differential equation $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + x^3 y = 2e^x$

$$y(1) = 2 \quad , \quad y'(1) = 5$$

has a unique solution (4 marks)