MACHAKOS UNIVERSITY

University Examinations 2021/2022 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR FIRST SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (STATISTICS AND COMPUTER PROGRAMMING)

BACHELOR OF SCIENCE (ACTUARIAL SCIENCE)

BACHELOR OF EDUCATION (SCIENCE)

SMA 460: STOCHASTIC PROCESSES

DATE: 31/8/2022 **TIME: 2.00-4.00 PM**

INSTRUCTION:

Answer question ONE and any other TWO questions

QUESTION ONE (COMPULSORY) (30 MARKS)

a) Define the following terms as used in statistics:

i. stochastic process (2 marks)

ii. wide-sense stationarity (1 mark)

b) State two conditions of a wide-sense stationary process. (2 marks)

c) Differentiate between a continuous-time process and a discrete-time process (2 marks)

d) Consider the Markov chain with state space {0,1} and transition probability matrix

$$P = \begin{bmatrix} 1 & 0 \\ 1/3 & 2/3 \end{bmatrix}$$

i. Show that the state 0 is recurrent (3 marks)

ii. Show that the state 1 is transient (3 marks)

e) i. Find the probability generating function (pgf) of the Poisson distribution

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$ (3 marks)

ii. Using the results of (i) above determine the mean of the Poisson distribution.(2 marks)

f) Let $\{X_i, i \geq 0\}$ be a simple random walk with $P(z_i=1) = p$ and $P(z_i = -1) = q = 1-p$ for all i, and let $X_i = \sum_{i=1}^{n} z_i \qquad i = 1, 2, \ldots$

and $X_0 = 0$

Now let the random process X(t) be defined by $X(t) = X_i,\ i \leq t < n+1$.

i. Describe X(t) (2 marks)

ii. Construct a typical sample function of X(t). (3 marks)

g) Show that the Poisson process $\{X(t)\}$ given by the probability distribution

$$P\{X(t) = k\} = \frac{e^{-\lambda t}(\lambda t)^k}{k!}, \qquad k = 0,1,2, \ldots$$

is not wide-sense stationary. (4 marks)

h) Given the matrix of transition probabilities

$$P = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Draw the state transition diagram. (3 marks)

QUESTION TWO (20 MARKS)

a) Define a counting process. (1 mark)

b) State three (3) conditions of a counting process. (3 marks)

c) Let $x_1, x_2, \ldots$ be independent Bernoulli random variables with $P(x_i=1) = p$ and $P(x_i=0) = q = 1-p$ for all i. The collection of random variables $\{X_i, i \geq 1\}$ is a random process, and it is called a Bernoulli process.

i. Describe the Bernoulli process. (2 marks)

ii. Construct a typical sample sequence of the Bernoulli process and plot it. (4 marks)

d) Consider the transition matrix

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix}$$

where α is the probability that it will rain tomorrow if it rains today and β is the probability that it will rain tomorrow if it does not rain today. Letting the state be 0 when it rains and 1 when it does not rain, determine the limiting probabilities, π_0 and π_1. (10 marks)

QUESTION THREE (20 MARKS)

a) Define a Poisson process giving the conditions that are necessary for a process to be called a Poisson process. (4 marks)

b) Patients arrive at a doctor's office according to a Poisson process with rate $\lambda = \frac{1}{20}$ minute. The doctor will not see a patient until at least three patients are in the waiting room.

i. Find the expected waiting time until the first patient is admitted to see the doctor. (5 marks)

ii. What is the probability that nobody is admitted to see the doctor in the first hour? (6 marks)

c) Suppose that people immigrate into an island at a Poisson rate $\lambda = 3$ per day.

i. What is the expected time until the 20^{th} immigrant arrives? (3 marks)

ii. What is the probability that the elapsed time between the 6^{th} and 7^{th} arrival exceeds 2 days? (2 marks)

QUESTION FOUR (20 MARKS)

a) Define the terms

i. Markov process

ii. Markov chain (4 marks)

b) Suppose that Mwadime's mood tomorrow depends on whether or not she is happy today. If she is happy today then she will be happy tomorrow with probability 0.7, and if she is not happy today she will be happy tomorrow with probability 0.4.

i. Define the states of the Markov chain (2 marks)

ii. Construct the matrix of transition probabilities (2 marks)

iii. Calculate the probability that Mwadime will be happy 2 days from today, and the probability that he will not be happy 4 days from today, given that he is happy today. (7 marks)

c) Consider the Markov chain consisting of the 4 states 0,1,2,3 and having transition probability matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i. Determine the classes of this Markov chain. (3 marks)

ii. Which is the absorbing state? Explain. (2 marks)

QUESTION FIVE (20 MARKS)

a) Briefly describe the following terms as used in stochastic processes:-

i. pure birth process

ii. death process (4 marks)

b) Find the mean size $\mu(t)$ of the population in the death process at time t. Show that $\mu(t)$ satisfies the differential equation

$$\frac{d}{dt}\mu(t) = -\mu.\,\mu(t)$$

where $p_{n_0}(0) = 1$. (6 marks)

c) Customers arrive at a watch repair shop according to a Poisson process at a rate of one per every 10 minutes, and the service time is an exponential random variable with mean 8 minutes.

i. Define the queuing system. (1 mark)

ii. Find the average number of customers L, the average time a customer spends in the shop W, and the average time a customer spends in waiting for service, W_q. (4 marks)

iii. Suppose that the arrival rate of the customers increases by 10%. Find the corresponding changes in L, W and W_q. (5 marks)