



MACHAKOS UNIVERSITY

University Examinations 2018/2019

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE

FOURTH YEAR SPECIAL/SUPPLEMENTARY EXAMINATION FOR

BACHELOR OF STATISTICS AND PROGRAMMING

SST402: BAYESIAN STATISTICS

DATE: 24/7/2019

TIME: 2:00 – 4:00 PM

INSTRUCTION: Answer Question One and any other Two Questions

QUESTION ONE

- a) Carefully outline the main stages of a typical Bayesian analysis procedure. (4 marks)
- b) Define the following terms as used in Bayesian statistics:
- i. Bayes estimator.
 - ii. Minimax estimator
 - iii. Highest posterior density interval (HPD) (6 marks)
- c) Let X be a sample of size n from a normal distribution with mean θ and variance 1. Consider estimating θ with squared error loss using two estimators:
- $$\hat{\theta}(x) = 2x \text{ and } \hat{\theta}_{MLE}(x) = x$$
- Determine which of the two estimators is inadmissible. (6marks)
- d) Differentiate between the following terms as used in Bayesian statistics:
- i. Confidence intervals and Credible intervals (4 marks)
 - ii. Prior distribution and posterior distribution (4 marks)
- e) Suppose $x_1, x_2, \dots, x_n$ each have an exponential distribution with parameter θ , and suppose that the prior for θ is an exponential distribution with parameter λ . Find the posterior distribution of θ . (6 marks)

QUESTION TWO (20 MARKS)

- a) Let $X_1, X_2, \dots, X_n$ be i.i.d random variables having a normal distribution with unknown mean μ and known variance δ^2 . Assuming a prior normal distribution for μ with mean μ_0 and variance δ_0^2 . Show that the posterior distribution for μ is normally distributed with mean μ_* and variance δ_*^2 where;

$$\mu_* = \delta_*^2 \left(\frac{\mu_0}{\delta_0^2} + \frac{n\bar{x}}{\delta^2} \right) \text{ and } \delta_*^2 = \left(\frac{1}{\delta_0^2} + \frac{n}{\delta^2} \right)^{-1}$$

(10 marks)

- b) Let $X_1, X_2, \dots, X_{10}$ be i.i.d random variables having a normal distribution with unknown mean θ and known variance 1. Assuming a prior normal distribution for θ with mean zero and variance 5. Let the sample mean be $\bar{x} = 1.873$
- i) Compute the posterior distribution of θ (6 marks)
- ii) Compute the 95% credible intervals for θ . (4 marks)

QUESTION THREE (20 MARKS)

- a) Define the following terms :
- i. Loss function. (2 marks)
- ii. Risk function (2 marks)
- b) Let $X_1, X_2, \dots, X_n$ be n independent observations from a population with density function $f(x/\theta)$ where θ is unknown parameter. Let $\delta(x)$ and $\lambda(\theta)$ denote the Bayes estimator of $g(\theta)$ and prior distribution of θ respectively. By considering a quadratic loss function of the form $L(\delta(x), \theta) = C(\theta)[\delta(x) - g(\theta)]^2$ where $C(\theta) \geq 0$, show that $\delta(\underline{x}) = E[g(\theta)/\underline{x}]$ (7 marks)

- c) Given $f(x/p) = \begin{cases} p(1-p)^{x-1}, & 0 \leq p \leq 1 \\ 0, & \text{otherwise} \end{cases}$

Suppose that the prior distribution of p is;

$$h(p) = \begin{cases} 1, & 0 \leq p \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the Bayes estimator of p with respect to the loss function defined as ;

$$L(\delta(x), p) = C(p)[\delta(x) - p]^2 \text{ where } C(p) \geq 0. \quad (9 \text{ marks})$$

QUESTION FOUR (20 MARKS)

- a) Let $X_1, X_2, \dots, X_n$ be n independent observations from

$$f(x|\theta) = \begin{cases} \theta^x (1-\theta)^{1-x}, & x = 0, 1 \\ 0 & , \text{ otherwise} \end{cases}$$

Let $L(\delta(x), \theta) = [\delta(x) - \theta]^2$. Show that the minimax estimator of θ for the squared error loss function is;

$$\delta(\underline{x}) = \frac{\bar{x}\sqrt{n}}{1+\sqrt{n}} + \frac{1}{2(1+\sqrt{n})} \quad \text{where} \quad \bar{x} = \frac{\sum x_i}{n} \quad (15 \text{ marks})$$

- b) Define conjugacy and explain why or why not, the beta prior is conjugate with respect to the negative Binomial likelihood. (5 marks)

QUESTION FIVE (20 MARKS)

- a) Suppose that $X_1, X_2, \dots, X_n / \theta \approx i.i.d$ Poisson (λ).

- What prior is conjugate for the Poisson likelihood? Give the distribution for λ along with any associated parameters. (2 marks)
- Calculate the posterior distribution of $\lambda / \underline{x}$ using your prior in (i). (4 marks)

- b) The numbers of sales of a particular item from an internet retail site in each of 20 weeks are recorded. Assume that, given the value of a parameter λ , these numbers are independent observations from the Poisson (λ) distribution.

Our prior distribution for λ is a gamma (α, β) distribution.

- Our prior mean and standard deviation for λ are 16 and 8 respectively. Find the values of α and β . (4 marks)
- The observed numbers of sales are as follows.
14, 19, 14, 21, 22, 33, 15, 13, 16, 19, 27, 21, 16, 25, 14, 23, 22, 17.
Find the posterior distribution of λ . (4 marks)

- Find a 95% posterior hpd interval for λ . (2 marks)

(Note: If $X \sim \text{gamma}(\alpha, \beta)$, i.e. $f(x) = k x^{\alpha-1} e^{(-\beta x)}$, then the mean of X is

$$E(X) = \alpha/\beta \quad \text{and the variance of } X \text{ is } \text{var}(X) = \alpha/\beta^2)$$

- c) Discuss briefly the following Markov Chains Monte Carlo methods (MCMC) methods.

- Gibbs sampling. (2 marks)
- Metropolis Hastings (2 marks)