



MACHAKOS UNIVERSITY

University Examinations 2020/2021 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECOND YEAR SPECIAL /SUPPLEMENTARY EXAMINATION FOR

BACHELOR OF SCIENCE IN MATHEMATICS

SMA 204: ALGEBRAIC STRUCTURES

DATE: 24/3/2021

TIME: 8.30-10.30 AM

INSTRUCTIONS TO CANDIDATES

Answer Question One and any other two questions

QUESTION ONE (30 MARKS)

- a) Explain the meaning of the following terms
- (i) a commutative binary operation on a set A , (2 marks)
 - (ii) an associative binary operation on a set A . (2 marks)
- b) Let $\mathbb{R} \setminus \{0\}$ be the set of all non-zero real numbers and let $\times$ be the usual multiplication of real numbers. Prove that $(\mathbb{R} \setminus \{0\}, \times)$ is a group. (5 marks)
- c) Define a relation $\sim$ on the set $\mathbb{Z}$ as follows: $a \sim b$ if and only if $a - b$ is a multiple of three. Determine if $\sim$ is an equivalence relation. (5 marks)
- d) Let $\oplus$ denote operation of addition modulo 5 on the set $\mathbb{Z}_5$. Construct the Cayley table for the operation $\oplus$ on $\mathbb{Z}_5$. (5 marks)
- e) Consider $\mathbb{Z}_{20}$ under multiplication modulo 20 (denoted by $\otimes$).
- (i) Is $\langle \mathbb{Z}_{20}, \otimes \rangle$ a group? Justify your answer. (3 marks)
 - (ii) Determine the elements of U_{20} and construct the corresponding Cayley table. (4 marks)
 - (iii) Let H be the cyclic subgroup of U_{20} generated by $13 \in U_{20}$. Determine the elements of H . (4 marks)

QUESTION TWO (20 MARKS)

- a) Let $\odot$ be a binary operation defined on $\mathbb{Z}$ by $a \odot b = ab - a - b$.
- (i) Show that $\odot$ is a commutative binary operation. (2 marks)
- (ii) Is $\odot$ an associative binary operation? Justify your answer. (2 marks)
- b) Let $\mathbb{Q}$ be the group of rational numbers under standard addition. Let F_7 be the subset of $\mathbb{Q}$ given by

$$F_7 = \left\{ x \in \mathbb{Q} \mid x = \frac{m}{7} \text{ for some } m \in \mathbb{Z} \right\}.$$

- Determine whether F_7 is a subgroup of $\mathbb{Q}$. (5 marks)
- c) Suppose that G is a group and $a \in G$ be an element such that $a^2 = a$. Prove that $a = e$ where e is the identity element in G . (2 marks)
- d) Let $M_2(\mathbb{Z})$ denote the group of all 2×2 matrices with integer entries under matrix addition. Let T be the subset of $M_2(\mathbb{Z})$ given by

$$T = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in M_2(\mathbb{Z}) \mid a, b \text{ and } c \text{ are even numbers} \right\}.$$

Show that T is a subgroup of $M_2(\mathbb{Z})$. (5 marks)

QUESTION THREE (20 MARKS)

- a) State the Lagrange's theorem. (2 marks)
- b) Let $G = \langle \mathbb{Z}, + \rangle$ be the group of integers under the standard addition.
- (i) Let K be the subset consisting of all odd numbers in $\mathbb{Z}$. Is K a subgroup of G ? Justify. (2 marks)
- (ii) Let H be the subset consisting of all even numbers in $\mathbb{Z}$. Prove that H is a subgroup of G . (3 marks)
- (iii) Determine all the distinct left cosets of H . (3 marks)
- c) Let G be a group
- (i) Show that if $a, b \in G$ are elements such that $a^2 b^2 = a^4 b$ then $ba = a^3$. (3 marks)
- (ii) Prove that if for all $a, b \in G$ then G is a commutative group. (3 marks)
- d) Suppose that G is a group of order 105 and suppose that H is a non-trivial subgroup of G . What are the possible values of $o(H)$? Justify your answer. (4 marks)

QUESTION FOUR (20 MARKS)

- a) Explain the meaning of the following terms:
- (i) a *right coset* of a subgroup, (2 marks)
 - (ii) the *kernel* of a group homomorphism. (2 marks)
- b) Let $U_{14} = \{1, 3, 5, 9, 11, 13\}$ be the group of invertible elements in $\mathbb{Z}_{14}$ under multiplication modulo 14. Let H be the cyclic subgroup of U_{14} generated by 13. Determine all the distinct left cosets of H . (6 marks)
- c) Let $G_1 = \langle \mathbb{R}, + \rangle$ be the group of real numbers under standard addition of real numbers and $G_2 = \langle \mathbb{R}^{+, \times} \rangle$ be the group of all positive real numbers under standard multiplication. Show that
- (i) the function $f: G_1 \rightarrow G_2$ defined by $f(x) = \exp(x)$ for any $x \in G_1$ is a group homomorphism. (Here, $\exp(x) = e^x$ is the usual exponential function). (5 marks)
 - (ii) the function $g: G_2 \rightarrow G_1$ defined by $g(a) = \ln(a)$ for any $a \in G_2$ is a group homomorphism. (Here $\ln(a)$ is the usual natural logarithm function). (5 marks)

QUESTION FIVE (20 MARKS)

- a) Explain the meaning of the following terms:
- i. an *integral domain*, (2 marks)
 - ii. the *characteristic* of a ring, (2 marks)
- b) Suppose that R is a set endowed with an additive binary $+$ and a multiplicative binary operation denoted by $\times$. What properties must the triple $(R, +, \times)$ satisfy in order for it to be a ring? (8 marks)
- c) Let $\mathbb{Q}$ denote the set of rational numbers. Define the set

$$\mathbb{Q}(\sqrt{3}) = \{a + b\sqrt{3} : a, b \in \mathbb{Q}\}$$

If we define addition and multiplication on $\mathbb{Q}(\sqrt{3})$ as follows

$$\begin{aligned}(a + b\sqrt{3}) \oplus (c + d\sqrt{3}) &= (a + c) + (b + d)\sqrt{3} \\ (a + b\sqrt{3}) \otimes (c + d\sqrt{3}) &= (a + b\sqrt{3})(c + d\sqrt{3})\end{aligned}$$

Show that $(\mathbb{Q}(\sqrt{3}), \oplus, \otimes)$ is a ring. (8 marks)