MACHAKOS UNIVERSITY

University Examinations 2018/2019
SCHOOL OF PURE AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS AND STATISTICS
THIRD YEAR SECOND SEMESTER EXAMINATIONS FOR
BACHELOR OF SCIENCE IN MATHEMATICS AND COMPUTER SCIENCE
BACHELOR OF SCIENCE IN MATHEMATICS
BACHELOR OF SCIENCE IN STATISTICS AND PROGRAMMING
BACHELOR OF EDUCATION – ARTS, SCIENCE, SPECIAL

SMA 303: RING THEORY

DATE:**24/04/2019** TIME: **2:00 – 4:00pm**

Answer Question One and any other two questions

Question One - (30 Marks)

(a) Explain the meaning of the following terms: **(6 marks)**

(i) a *commutative ring,*

(ii) a *zero divisor* in a ring R,

(iii) a *left ideal* in a ring R.

(b) Prove that if R is a ring and $a, b, c, d \in R$ then $(a + b)(c + d) = ac + ad + bc + bd$. **(4 marks)**

(c) Let $\mathbb{Q}[5]$ denote the set $\mathbb{Q}[5] = \{x + y\sqrt{5}: x, y \in \mathbb{Q}\}$. Show that the binary operation $\otimes$ defined by

$$(x + y\sqrt{5}) \otimes (z + w\sqrt{5}) = (xz + 5yw) + (xw + yz)\sqrt{5}$$

is an associative binary operation. **(5 marks)**

(d) Let $M_2(\mathbb{R})$ be the ring of all 2×2 matrices with real entries. Consider the subset $D \subset M_2(\mathbb{R})$ given by

$$D = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

i.e. D is the subset containing diagonal matrices. Prove that D is a subring of $M_2(\mathbb{R})$. **(5 marks)**

(e) Let $(\mathbb{Z}_{12}, \oplus, \otimes)$ denote the ring of integers modulo 12 under addition and multiplication modulo 12. Let I be the *principal ideal* of $\mathbb{Z}_{12}$ generated by 4.

(i) List the all elements of I. **(4 marks)**

(ii) Identify all the elements of the *quotient ring* $\mathbb{Z}_{12}/I$. **(6 marks)**

Question Two - (20 Marks)

(a) Explain the meaning of the following terms **(4 marks)**

(i) a *unit* in a ring R,

(ii) a *division ring*

(b) Let $(\mathbb{Z}_{21}, \oplus, \otimes)$ be the ring of integers modulo 21 under addition and multiplication modulo 21.

(i) Is $14 \in \mathbb{Z}_{21}$ a unit. Justify your answer. **(2 marks)**

(ii) Is $17 \in Z_{21}$ a zero divisor? Justify your answer. **(2 marks)**

(c) Let R be an integral domain. Prove that the characteristic of R is either 0 or a prime number p. **(6 marks)**

(d) Suppose that $\varphi: A \to B$ is a surjective ring homomorphism. Prove that

(i) if $I \subset A$ is a prime ideal then the image $\varphi(I)$ is a prime ideal of B, **(3 marks)**

(ii) if $J \subset B$ is a prime ideal then $\varphi^{-1}(J) = \{a \in A | \varphi(a) \in J\}$ is also a prime ideal. **(3 marks)**

Question Three - (20 Marks)

(a) Explain the meaning of the following terms **(4 marks)**

(i) a *subring* S in a ring R,

(ii) a *ring homomorphism* $\varphi: A \to B$.

(b) Let R be a ring and $Z(R) = \{x \in R: xy = yx \text{ for all } y \in R\}$ i.e. $Z(R)$ is the center of R. Prove that $Z(R)$ is a subring of R. **(5 marks)**

(c) Suppose that A and B are rings. Prove that if $\varphi: A \to B$ is a ring homomorphism then

(i) $\varphi(A)$ is a subring of B, **(4 marks)**

(ii) $Ker\ \varphi$ is an ideal of A, **(4 marks)**

(iii) if A is a commutative ring then $\varphi(A)$ is commutative. **(3 marks)**

Question Four - (20 Marks)

(a) Explain the meaning of the following terms **(4 marks)**

(i) a *prime ideal* in a ring R,

(ii) a *maximal ideal* in a ring R.

(b) Suppose that U and V are ideals of a ring R.

(i) prove that $U \cap V$ is also an ideal of R, **(4 marks)**

(ii) let $U + V = \{u + v : u \in U \text{ and } v \in V\}$ R. Prove that the set $U + V$ is also an ideal of R. **(4 marks)**

(c) Suppose that $\varphi: A \to B$ is a ring homomorphism and let $a \in A$.

(i) Show that $\varphi(a^n) = \varphi(a)^n$. **(4 marks)**

(ii) Show that if a is a unit in A then $\varphi(a)$ is a unit in B and $\varphi(a)^{-1} = \varphi(a^{-1})$. **(4 marks)**

Question Five - (20 Marks)

(a) Consider the map $\psi: \mathbb{R}^2 \to \mathbb{R}$ defined by $\psi(x, y) = x + y$. Determine if ψ a ring homomorphism. If so, determine the kernel of ψ. Justify your answer. **(4 marks)**

(b) Let R be a commutative ring.

(i) Prove that $\{0\}$ and R are the only ideals of R if and only if R is a field. **(3 marks)**

(ii) Using (i) above or otherwise, prove that the ideal $\{0\}$ is maximal if and only if R is a field. **(3 marks)**

(c) Let $\mathbb{Q}[\sqrt{7}]$ denote the set $\{x + y\sqrt{7} | x, y \in \mathbb{Q}\}$. Define the binary operations $\oplus$ and $\otimes$ on $\mathbb{Q}[\sqrt{7}]$ as follows

$$\begin{aligned}(x + y\sqrt{7}) \oplus (z + w\sqrt{7}) &= (x + y) + (y + w)\sqrt{7} \\ (x + y\sqrt{7}) \otimes (z + w\sqrt{7}) &= (xz + 7yw) + (xw + yz)\sqrt{7}\end{aligned}$$

(i) Prove that $\mathbb{Q}[\sqrt{7}]$ is a subring of $\mathbb{R}$. **(4 marks)**

(ii) Determine if every non-zero elements of $\mathbb{Q}[\sqrt{7}]$ is invertible. **(6 marks)**