

UNIVERSITY MACHAKOS
UNIVERSITY EXAMINATION, 2018/2019
SUPPLEMENTARY EXAMINATION
EXAMINATIONS FOR THE DEGREE OF BACHELOR OF
SCIENCE IN STATISTICS AND PROGRAMMING
SMA 466: MULTIVARIATE ANALYSIS

Attempt Question 1 and any other 2 questions

Date:

Time:

QUESTION ONE (30 Marks)

(a). Suppose $\underline{X} = (x_1 \ x_2 \ x_3)^T$

$$\underline{\mu} = (\mu_1, \mu_2, \mu_3)^T \text{ and } \Sigma = \begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & 0 \\ \rho^2 & 0 & 1 \end{pmatrix}$$

If $\underline{X}$ is $N_3[\underline{\mu}, \Sigma]$

Show that $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ given x_3 has a multivariate normal distribution with

mean $\underline{r}$ and covariance matrix V

where

$$\underline{r} = (\mu_1 + \rho^2(x_3 - \mu_3), \mu_2)^T$$

and

$$V = \begin{pmatrix} 1 - \rho^4 & \rho \\ \rho & 1 \end{pmatrix} \quad (7 \text{ marks})$$

(b). (i). Describe the Wishart distribution,

$$W_p = [\Sigma, n]$$

- (ii). Let the random vector variable $\underline{Z}_{p \times 1}$ have the multivariate Normal distribution with mean $\underline{Q}_{p \times 1}$, and covariance matrix $\underline{I}_{p \times p}$ where $\underline{I}_{p \times p}$ is a p by p identity matrix.

Determine the distribution of $\underline{Z}^T \underline{Z}$ and hence or otherwise show that

$$W_p [\underline{I}_{p \times p}, \underline{P}] = \chi^2_{(p)}$$

(8 marks)

- (c). Let $\underline{X}$ be a $p \times 1$ random vector and $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n$ be a random sample of size n from the population.

Describe

(i). $\underline{X}_r$ for some $r = 1, 2, \dots, n$

(ii). The sample mean vector $\bar{\underline{X}}$

(iii). The sample dispersion matrix $\underline{S}$.

(iv). The data matrix $\underline{X}$

(8 marks)

- (d). Using the notations in part (a), show that

(i). $E(\bar{\underline{X}}) = \underline{\mu}$ and

(ii). $E(\underline{S}) = \frac{n-1}{n} \underline{\Sigma}$ where $\underline{\mu}$ and $\underline{\Sigma}$ denote the population mean vector and variance-covariance vector respectively.

(7 marks)

QUESTION TWO (20 marks)

(a). Let $\underline{X}$ be a p-component random vector which has the multivariate distribution, $N_p[\underline{\mu}, \underline{\Sigma}]$. Show that for the sample mean vector $\bar{\underline{X}}$ and sample variance S;

$$(i). \bar{\underline{X}} \sim N_p \left[\underline{\mu}, \frac{\underline{\Sigma}}{n} \right] \quad (4 \text{ marks})$$

$$(ii). S \sim W_p \left[\frac{\underline{\Sigma}}{n}, n-1 \right] \quad (4 \text{ marks})$$

$$(iii). \frac{n-p}{p} \left(\bar{\underline{X}} - \underline{\mu} \right)^T S^{-1} \left(\bar{\underline{X}} - \underline{\mu} \right) \sim F(p, n-1) \quad (4 \text{ marks})$$

(b). Let $\underline{Y}$ be a random vector with mean $\underline{\mu}$ and covariance matrix $\underline{\Sigma} = I_{p \times p}$. Show that if $\underline{Y}$ is $N_p[\underline{\mu}, \underline{\Sigma}]$, then $E \{ (\underline{Y} - \underline{\mu})^T (\underline{Y} - \underline{\mu}) \} = p$

(8 marks)

QUESTION THREE (20 marks)

(a). Show that if $\underline{Y}$ is $N_p[\underline{\mu}, \underline{\Sigma}]$ then an rx1 subvector of $\underline{Y}$ has an r-variate normal distribution with the same means, variances and covariances as in the original p-variate normal distribution.

(6 marks)

(b). Using part (a), or otherwise, Show that any individual variable Y_i in $\underline{Y}$ is distributed as $N_p[\mu_i, \sigma_{ii}]$, where $\mu_i = E(Y_i)$, $\sigma_{ii} = V(Y_i)$ (6 marks)

(c). Show that if $\underline{Y}$ and $\underline{X}$ are jointly multivariate normal with $\underline{\Sigma}_{yx} \neq 0$, then the conditional distribution of $\underline{Y}$ given as $\underline{X}$, $f(\underline{Y}/\underline{X})$ is multivariate normal with mean vector and covariance matrix.

$$E(\underline{Y}/\underline{X}) = \underline{\mu}_y + \underline{\Sigma}_{yx} \underline{\Sigma}_{xx}^{-1} (\underline{X} - \underline{\mu}_x)$$

$$V(\underline{Y}/\underline{X}) = \underline{\Sigma}_{yy} - \underline{\Sigma}_{yx} \underline{\Sigma}_{xx}^{-1} \underline{\Sigma}_{yx}$$

respectively, given corresponding covariance matrices (8 marks)

QUESTION FOUR (20 marks)

Let the random vector $\underline{V}$ be $N_4[\underline{\mu}, \underline{\Sigma}]$ where $\underline{\mu} = (2 \ 5 \ -2 \ 1)^T$,

and

$$\Sigma = \begin{pmatrix} 9 & 0 & 3 & 3 \\ 0 & 1 & -1 & 2 \\ 3 & -1 & 6 & -3 \\ 3 & 2 & -1 & 7 \end{pmatrix}$$

Let $\underline{Y} = (Y_1 \ Y_2 \ X_1 \ X_2)$ (6 marks)

(a). Determine the distribution for $(Y_1 \ Y_2)^1 / (X_1 \ X_2)^1$

(b). Hence or otherwise, calculate

(i). $\text{Cov}(X_1, X_2)$

(ii). $\text{Cov}((X_1 \ X_2)^1 / (Y_1 \ Y_2)^1)$

(iii) Comment on the results in b(i) and b(ii) (8 marks)

(c). Find the partial correlation matrix

($\text{corr}((X_i \ X_j)^1 / (Y_1 \ Y_2)^1)$ for $i, j=1, 2$) (6 marks)

QUESTION FIVE (20 marks)

Consider a random vector $x^1 = (x_1 \ x_2)$. Assume x has $N^2[\mu, \Sigma]$ distribution.

A sample of size $n = 25$ has $\bar{X} = (185.72, 183.84)^1$

(a). Suppose Σ is unknown and the sample dispersion matrix

$$S = \begin{pmatrix} 91.481 & 66.875 \\ 66.875 & 96.775 \end{pmatrix}$$

Test $H_0: \Sigma = \Sigma_0$ where $\Sigma_0 = \begin{pmatrix} 100 & 0 \\ 0 & 100 \end{pmatrix}$

Assume 5% level of significance. (10 marks)

(b). Suppose now $\Sigma = \begin{pmatrix} 100 & 0 \\ 0 & 100 \end{pmatrix}$ while the mean vector $\underline{\mu}$ is unknown.

Test

$$H_0: \mu = \mu_0$$

where $\mu_0 = (182 \ 182)^1$ at 5% level of significance. (10 marks)

- (a). Suppose that B is a $k \times p$ matrix of constants. A is a $p \times p$ symmetric matrix of constants, and $\underline{Y}$ is distributed as $N_p[\underline{\mu}, \Sigma]$.

Show that $B\underline{Y}$ and $\underline{Y}^T A \underline{Y}$ are independent if and only if $B \Sigma A = 0$

(10 marks)

- (b). Let A and B be symmetric matrices of constants. If $\underline{Y}$ has $N_p[\underline{\mu}, \Sigma]$ distribution, show that $\underline{Y}^T A \underline{Y}$ and $\underline{Y}^T B \underline{Y}$ are independent if and only if

$$A \Sigma B = 0 \quad (10 \text{ marks})$$

QUESTION FOUR (20 marks)

- (a). Let $\underline{X}$ be the p -component vector describe in Question ONE. If

(i). $\underline{X} \sim N_p\left[\underline{\mu}, \frac{\Sigma}{n}\right]$ (4 marks)

(ii). $S \sim W_p\left[\frac{\Sigma}{n}, n-1\right]$ (4 marks)

(iii). $\frac{n-p}{p}(\bar{X} - \underline{\mu})^T S^{-1}(\bar{X} - \underline{\mu}) \sim F(p, n-1)$ (4 marks)

- (b). Let $\underline{Y}$ be a r.u. with mean $\underline{\mu}$ and covariance matrix Σ . Show that if $\underline{Y}$ is $N_p[\underline{\mu}, \Sigma]$, then $\text{Var}(\underline{Y}^T A \underline{Y}) = 2 \text{trace}\{(A \Sigma)^2\} + 4 \underline{\mu}^T A \underline{\mu}$

(8 marks)