MACHAKOS UNIVERSITY

University Examinations 2018/2019

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE

THIRD YEARSPECIAL/SUPPLEMENTARY EXAMINATION FOR

BACHELOR OF SCIENCE IN MATHEMATICS AND COMPUTER SCIENCE
BACHELOR OF SCIENCE IN STATISTICS AND PROGRAMMING
BACHELOR OF SCIENCE IN MATHEMATICS
BACHELOR OF ECONOMICS AND STATISTICS
BACHELOR OF EDUCATION
BACHELOR OF ARTS

SMA 331: DYNAMICS II

DATE: 25/7/2019 **TIME: 8:30 – 10:30 AM**

Instructions to the Candidate:

Answer question one (**compulsory**) and any other **two** questions

QUESTION ONE (30 MARKS)

a) Let $\boldsymbol{F}(r) = -\boldsymbol{V}'(r)$ for some function $\boldsymbol{V}$. Prove that the central force $\boldsymbol{F}(r) = \boldsymbol{F}(r)\hat{\boldsymbol{\imath}}$ is conservative with the potential function $\boldsymbol{V}'(r)$. (3 marks)

b) A particle of mass m moving with a velocity $\boldsymbol{v}$ has an angular momentum $\boldsymbol{L} = m(\boldsymbol{r} \times \boldsymbol{v})$ at any position $\boldsymbol{r}$. Given that the angular momentum $\boldsymbol{L}$ points in the z-direction. Using polar coordinates, show that;

i) The magnitude of angular momentum is given by $L = mr^2\dot{\theta}$. (4 marks)

ii) The motion of the particle is governed by the radial equation

$\ddot{r} - r\dot{\theta}^2 = -\frac{V'(r)}{m}$ (4 marks)

c) A small body of mass m slides on a rod which is the chord of a circular wheel. The wheel rotates about its centre O with a clockwise angular velocity $\omega = 4\, rad/sec$ and a clockwise angular acceleration $\dot{\omega} = 5\, rad/sec^2$. The body m has a constant velocity of 1.83 m/s to the right, relative to the wheel. Given that $\rho = 0.4575\ m$ and $R = 0.915\ m$, determine the absolute acceleration of the mass. (5 marks)

d) Given that the function $y(x)$ has a graph passing through two points $x_1$and x_2. By minimizing the integral;

$$I = \int_{x_1}^{x_2} [1 + y'(x)^2]^{1/2}\, dx,$$

Show that $y(x)$ is a linear function defining the shortest distance between the two points. (5 marks)

e) Consider a single particle of mass m, moving subject to a force field. If the vector from a fixed origin to the particle at a time t is denoted by $\boldsymbol{r}$. Using the Newton's laws of motion, derive the Hamilton's principle in its most general form. (5 marks)

f) A rigid body performs plane motion such that all points of the body move parallel to the plane. Considering three points *A, B* and *C* in a moving coordinate system such that *C* is considered to be the origin of the moving coordinate system (x, y, z) with reference to a fixed coordinate system and the velocities at the other two points *A* and *B* are known, show that the angular velocity of the body is given by;

$$\boldsymbol{\omega} = \frac{V_A}{\rho_A} = \frac{V_B}{\rho_B}$$

Where V_A and V_B are the velocities of the body at *A* and *B* respectively and ρ_A and ρ_B are the displacements of the body from *C* to *A* and *B* respectively. (4 marks)

QUESTION TWO (20 MARKS)

A planet of mass m orbiting around a star of mass M located at the origin experiences a central force $\boldsymbol{F} = -\frac{GMm}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}$ and has an energy $E = \frac{1}{2}mv^2 - \frac{K}{r}$, where $K = GMm$;

a) Using Runge' vector $\boldsymbol{R} = \hat{\boldsymbol{r}} + \frac{\boldsymbol{L}}{\boldsymbol{K}}(\hat{\boldsymbol{k}} \times \boldsymbol{v})$, show that R remains constant in time. (3 marks)

b) Given that $\hat{\boldsymbol{k}} \times \boldsymbol{v} = \dot{r}\hat{\boldsymbol{\theta}} - r\dot{\boldsymbol{\theta}}\hat{\boldsymbol{r}}$;

i) Derive an expression for the eccentricity e of the orbit of the planet.

(4 marks)

ii) Show that the radius of the planet from the origin is given by

$r = \frac{L^2}{Km}\left(\frac{1}{1-e\cos\theta}\right)$ (4 marks)

iii) By letting $\beta = \frac{L^2}{Km}$, define;

I. Aphelion distance. (1 marks)

II. Perihelion distance. (1 marks)

c) Prove that the orbits of planets are ellipses with the sun at one focus and are centered at $(ea, 0)$ having the semi-major axis a and semi-minor axis $b = \sqrt{1-e^2a}$.

(4 marks)

d) Prove that the period T of a revolution of orbit is proportional to $a^{3/2}$. (3 marks)

QUESTION THREE (20 MARKS)

a) Considering a moving coordinate system; Show that the general expression for the acceleration of a point refereed to a coordinate system which itself is moving is given by;

$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \ddot{\boldsymbol{\rho}}\boldsymbol{r} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}\boldsymbol{r}$ (10 marks)

b) Two concentric circles of radius r and R rotate about the sane fixed centre O. The constant angular velocity of the large disk measured in a fixed system is given by $\boldsymbol{\Omega}$. The constant angular velocity of the small disk relative to the large disk $\boldsymbol{\omega}$. Find the acceleration of a point A, on the circumference of the small disk and indicate the acceleration of coriolis. (5 marks)

QUESTION FOUR (20 MARKS)

a) Given that the function $y(x)$ has a graph passing through two points x_1 and x_2. By minimizing the integral;

$$I = \int_{x_1}^{x_2} 2\pi y(x)[1 + y'(x)^2]^{1/2}\, dx,$$

Determine the minimal surface of revolution passing through the given points.

(10 marks)

b) Obtain the Lagrange equation for a point mass m suspended by an inextensible string of length l. (8 marks)

QUESTION FIVE (20 MARKS)

a) Consider a one-dimensional simple harmonic oscillator whose kinetic energy $T = \frac{1}{2}m\dot{x}^2$ and potential $V = \frac{1}{2}Kx^2$. Using Hamilton-Jacobi theory, obtain the position $x(t)$ and momentum $\rho(t)$ of the oscillator. (15 marks)

b) The centre of mass of a rigid body of mass $50\ lb\ sec^2/ft$ has a velocity of magnitude40 ft/sec. At a given instant the moment of momentum of the body about the centre of mass is $H_c = 1500\boldsymbol{i} + 1000\boldsymbol{j} + 1200\boldsymbol{k}\ lb\ ft\ sec$. The inertia integrals about the same coordinate axes are;

$$I = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 40 & -20 \\ 0 & -20 & 30 \end{bmatrix} lb\ ft\ sec^2$$

Find the angular velocity and kinetic energy of the body at a given instant.

(5 marks)