# MACHAKOS UNIVERSITY

**University Examinations for 2020/2021**

**SCHOOL OF PURE AND APPLIED SCIENCES**

**DEPARTMENT OF MATHEMATICS AND STATISTICS**

**FOURTH YEAR FIRST SEMESTER EXAMINATION FOR**

**BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)**

**BACHELOR OF SCIENCE (STATISTICS AND PROGRAMMING)**

**BACHELOR OF EDUCATION (SCIENCE)**

**SMA 404: COMPLEX ANALYSIS II**

**DATE: 11/8/2021** **TIME: 8:30 – 10:30 AM**

**INSTRUCTION:** *Answer Question **ONE** which is compulsory and any other **TWO** Questions*

**QUESTION ONE (COMPULSORY) (30 MARKS)**

a) State without proof the *Cauchy's residue theorem.* (2 marks)

b) Determine the poles of the function $f(z) = \frac{z+2}{z^2(z^2+9)}$ stating clearly the order of each pole. (4 marks)

c) Use the Cauchy's residue theorem to evaluate the following integral

$$\oint_C \frac{5z-3}{z^2(z-1)}dz$$

where $C$ is the circle $|z| = 2$. (5 marks)

d) Consider the conformal transformation defined by $w = z^2$. Compute:

(i) the coefficient of magnification at $z_0 = 3 + i$, (4 marks)

(ii) the angle of rotation at $z_0 = 3 + i$, (4 marks)

(iii) the critical points of the transformation. (2 marks)

e) Consider the function $u(x, y) = x^3 - 3xy^2$

(i) Prove that $u(x, y)$ is a harmonic function. (3 marks)

(ii) Use the *harmonic conjugate theorem* to determine the harmonic conjugate of the $u(x, y)$. (6 marks)

**QUESTION TWO ((20 MARKS)**

a) Explain the meaning of the following terms:

(i) *Conformal transformation*, (2 marks)

(ii) *Isogonal transformation*. (2 marks)

b) Determine whether the function $f(z) = z^2 + i$ satisfy the Schwarz reflection principle.

(6 marks)

c) Compute the residues of the function $f(z) = \frac{1}{z(z+1)}$ at all of its poles. (4 marks)

d) Compute the Cauchy's principal value of the following indefinite integral

$\int_{\infty}^{\infty} \frac{x^2}{(x^2+4)^2} dx.$ (6 marks)

**QUESTION THREE (20 MARKS)**

a) Explain the meaning of the term *analytic continuation*. (2 marks)

b) Compute the Taylor Series for the function

$$f(z) = \sin(z)$$

about the point $a = \frac{\pi}{4}$. (4 marks)

c) Evaluate the integral $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$. (4 marks)

d) Prove that the series $g(z) = \frac{1}{1+i} \sum_{n=0}^{\infty} \left(\frac{z+i}{1+i}\right)^n$ is an analytic continuation of the function $f(z) = \left(\frac{1}{1-z}\right) = \sum_{n=0}^{\infty} z^n$ and sketch the region where the two functions are analytic.

(7 marks)

e) State without proof the Schwarz reflection principle. (3 marks)

**QUESTION FOUR (20 MARKS)**

a) Explain the meaning of the following terms:

(i) *harmonic function* $h(x, y)$, (2 marks)

(ii) *harmonic conjugate* of a function $h(x, y)$. (2 marks)

b) Evaluate the contour integral $\oint_C \frac{(2z-1)}{z(z+1)(z-3)} dz$ where $C$ is given to be the circle $|z| = 2$.

(5 marks)

c) Determine the Laurent series for the function

$$f(z) = \frac{1}{1+z}$$

which are valid where $1 < |z| < \infty$. (5 marks)

d) Prove that for closed polygons, the sum of the exponents

$$\frac{\alpha_1}{\pi} - 1, \frac{\alpha_2}{\pi} - 1, \cdots \frac{\alpha_n}{\pi} - 1$$

in the Schwarz-Christoffel transformation is $-2$. (6 marks)

**QUESTION FIVE (20 MARKS)**

a) Show that the transformation $w = \sin z$ is conformal at all points except $z = \frac{\pi}{2} + n\pi$ for all $n \geq 0$. (4 marks)

b) Consider the transformation given by $w = z^2$.

(i) Determine the image of the triangular region on the $z$-plane bounded by the lines $x = 1, y = 1$ and $x + y = 1$. (5 marks)

(ii) Verify conformality at the point $z = 1$ by considering the intersection of the lines $x = 1$ and $x + y = 1$. (4 marks)

c) State without proof the *Schwarz-Christoffel transformation* theorem. (3 marks)

d) If $w = \infty$ is mapped onto a vertex $z_n$ on the $z$-plane, prove that the term containing $(w - u_n)$ in the Schwarz-Christoffel transformation can be replaced by 1. (4 marks)