MACHAKOS UNIVERSITY

University Examinations 2016/2017

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR FIRST SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS AND STATISTICS)

BACHELOR OF SCIENCE (MATHEMATICS)

BACHELOR OF SCIENCE (COMPUTER SCIENCE)

BACHELOR OF EDUCATION

SMA 402: FIELD THEORY

DATE: 26/7/2017 **TIME: 2:00 – 4:00 PM**

INSTRUCTIONS

Answer question ONE (Compulsory) and any other TWO questions

QUESTION ONE: 30 MARKS (COMPULSORY)

a) Define a Field. (2 marks)

b) Using relevant examples differentiate between a Field of characteristic p where p is a prime integer and a Field of characteristic zero. (3 marks)

c) Consider the finite field Z_7.Write down all the elements of this field. Identify all the generators of the cyclic group formed by the non-zero elements. (3 marks)

d) Show that $Q(\sqrt{2}) = \{a + b\sqrt{2}: a, b \in Q\}$ is a Field. (4 marks)

e) Consider the polynomial in $p(x) = x^2 + x + 1$ in $Z_2\,[x]$

i. Describe what is meant by an irreducible polynomial. (2 marks)

ii. Verify that the polynomial $p(x)$ is irreducible in Z_2 (1 mark)

iii. Let $Z_2[\alpha]$ be the extension field of Z_2 where α is algebraic over Z_2. In terms of α list and simplify all the elements of $Z_2[\alpha]$. Construct addition and multiplication tables of $Z_2[\alpha]$ then verify that $Z_2[\alpha]$ is a field. (9 marks)

iv. Find **irr**(α, Q) and **deg**$(\alpha, Q)$for the given algebraic number $\alpha \in C$over Q where $\alpha = \sqrt{2} + \boldsymbol{i}$ (3 marks)

v. Find a basis for the extension field $Q(\sqrt{2} + \boldsymbol{i})$over Q (3 marks)

QUESTION TWO (20 MARKS)

a) Give definitions of the following terms giving examples in each:

i. An algebraic element α of an extension field E of a field F. (2 marks)

ii. A transcentental element α of an extension field E of a field F. (2 marks)

b) Let $f(x) = x^4 - 2x^2 - 3$. Over the field Q.

i Find the zeros of the polynomial $f(x)$. (4 marks)

ii Find the splitting field F of the polynomial $f(x)$. (2 marks)

iii Exhibit a tower of fields in (ii) above (2 marks)

iv Find the index of Fin Q. (2 marks)

c) i. Describe what is meant by intersection of subfields: Give an example. (3 marks)

ii. Prove that $x^2 - 3$ is irreducible over $Q(\sqrt[3]{2})$ (3 marks)

QUESTION THREE (20 MARKS)

a) i Show that the polynomial x^2+ 1 is irreducible in $Z_3[x]$ (2marks)

ii Let β be a zero of the polynomial x^2+ 1 in the extension field $Z_3[\beta]$.
List down the nine elements of $Z_3[\beta]$.Give the addition and multiplication tables.
Justify that $Z_3[\beta]$ is the finite field $GF(3^2)$ (10 marks)

b) Show that a field F is algebraically closed if and only if every non constant polynomial in $F[x]$ factors in $F[x]$ into linear factors. (5 marks)

c) Find the degree and the basis for the field extension $Q(\sqrt[3]{2}\ ,\ \sqrt{3}\,)$ over Q (3 marks)

QUESTION FOUR (20 MARKS)

a) Show that if E is a finite extension field of a field F and K is a finite extension field of E then K is a finite extension of F and $[K:F] = [K:E][E:F]$ (10 marks)

b) Is the polynomial $p(x) = x^3 + 2x + 3$ in $Z_5[x]$ irreducible? Why? Express it as a product of irreducible polynomials in $Z_5[x]$. (5 marks)

c) Find all the generators of the cyclic multiplicative group of units of the field Z_{17} .

(5 marks)

QUESTION FIVE (20 MARKS)

a) i Show that it is impossible by straight edge and compass alone to trisect an angle of 60^0 (4 marks)

ii Show that it is impossible by straight edge and compass alone to duplicate the cube. (4 marks)

b) Write $F_3 = \{0,1,2\}$, the finite field of order 3 with arithmetic performed modulo 3. Let $\alpha = \sqrt{2}$.

i Is $2 + \alpha$ a square in $F_3(\alpha)$. (3 marks)

ii Let $\beta = \sqrt{2 + \sqrt{2}}$,calculate the degree of the extension $F_3(\beta): F_3(\alpha)$ (3 marks)

iii Calculate the degree of the extension $F_3(\beta): F_3$ (3 marks)

iv Justify that the set $\{a + b\sqrt{2} + c\sqrt{3} : a, b, c \in Q\}$ is a field. (3 marks)