MACHAKOS UNIVERSITY

University Examinations 2022/2023 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECOND YEAR SECOND SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)

BACHELOR OF SCIENCE (STATISTICS AND PROGRAMMING)

BACHELOR OF SCIENCE (MATHEMATICS)

SMA 204: ALGEBRAIC STRUCTURES

DATE: 18/4/2023 **TIME: 8:30 – 10:30 AM**

INSTRUCTIONS: Answer Question One and any other two questions

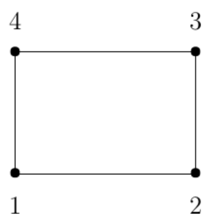
QUESTION ONE (COMPULSORY) (30 MARKS)

a) Explain the meaning of the following terms:

(i) a *group homomorphism* from a group G_1 to a group G_2, (3 marks)

(ii) an *invertible* element in a ring R (3 marks)

b) Consider a non-square rectangle as given in the diagram below



(i) Determine all the symmetries of this diagram. (4 marks)

(ii) Construct the Cayley for the associated binary operation. 4 marks)

c) Consider $\mathbb{Z}_{18}$ under multiplication modulo 18 (denoted by $\otimes$).

(i) Determine the elements of U_{18} and construct the its Cayley table. (6 marks)

(ii) Let H be the cyclic subgroup of U_{18} generated by $7 \in U_{18}$. Determine the elements of H. (5 marks)

d) Prove that every cyclic group G is commutative. (5 marks)

QUESTION TWO - (20 MARKS)

a) Let $\odot$ be a binary operation defined on $\mathbb{Q}$ by $a \odot b = \frac{a+b}{3}$.

(i) Show that $\odot$ is a commutative binary operation. (2 marks)

(ii) Is $\odot$ an associative binary operation? Justify your answer. (3 marks)

b) Let G be a group.

(i) State the cancellation laws in group theory. (4 marks)

(ii) Prove that for any $a, b \in G$ then $(ab)^{-1} = b^{-1}a^{-1}$ (4 marks)

c) Let a be an element in a group G such that $a^2 = a$. Prove that $a = e$ where e is the identity element in G. (2 marks)

d) Let $M_2(\mathbb{Z})$ be the group of all 2×2 matrices under usual matrix addition with integer entries. Let D be the subset of $M_2(\mathbb{Z})$ given by

$$D = \left\{ A \in M_2(\mathbb{R}) \text{ such that } A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \text{ for some } a, b \in \mathbb{Z} \right\}$$

Determine whether D is a subgroup of $M_2(\mathbb{Z})$. (5 marks)

QUESTION THREE - (20 MARKS)

a) State the Lagrange's theorem. (2 marks)

b) Let $G = \langle \mathbb{Z}, + \rangle$ be the group of integers under the standard addition.

(i) Let H be the subset consisting of all integers that are divisible by 5. Prove that H is a subgroup of G. (4 marks)

(ii) Determine all the distinct left cosets of H. (4 marks)

c) Let G be a group. Show that if $a, b \in G$ are elements such that $a^2b^2 = a^4b$ then $ba = a^3$. (4 marks)

d) Suppose that G is a group of order 1089 and suppose that H is a non-trivial subgroup of G. Determine are the possible values of $o(H)$. Justify your answer. (6 marks)

QUESTION FOUR (20 MARKS)

a) Explain the meaning of the following terms:

(i) a *left coset* of a subgroup, (2 marks)

(ii) the *kernel* of a group homomorphism. (2 marks)

b) Let $f: G_1 \to G_2$ be a group homomorphism and let e_1 and e_2 denote the identity elements in G_1 and G_2 respectively. Prove that

(i) $f(e_1) = e_2$, (4 marks)

(ii) $f(x^{-1}) = (f(x))^{-1}$ for any $x \in G_1$. (4 marks)

c) Let $\langle \mathbb{R}^*, + \rangle$ be the group of non-zero real numbers under usual multiplication. Determine whether the following functions $\phi: \mathbb{R}^* \to \mathbb{R}^*$ are group homomorphisms:

(i) $\phi(x) = x^2$ for all $x \in \mathbb{R}^*$. (4 marks)

(ii) $\phi(x) = 2^x$ for all $x \in \mathbb{R}^*$. (4 marks)

QUESTION FIVE - (20 MARKS)

a) Explain the meaning of the following terms:

(i) an *integral domain*, (2 marks)

(ii) the *characteristic* of a ring, (2 marks)

b) Suppose that R is a set endowed with an additive binary $+$ and a multiplicative binary operation denoted by $\times$. What properties must the triple $(R, +, \times)$ satisfy in order for it to be a ring? (8 marks)

c) Let $\mathbb{Q}$ denote the set of rational numbers. Define the set

$$\mathbb{Q}[\sqrt{3}] = \{a + b\sqrt{3} : a, b \in \mathbb{Q}\}$$

If we define addition and multiplication on $\mathbb{Q}[\sqrt{3}]$ as follows

$$(a + b\sqrt{3}) \oplus (c + d\sqrt{3}) = (a + c) + (b + d)\sqrt{3}$$
$$(a + b\sqrt{3}) \otimes (c + d\sqrt{3}) = (a + b\sqrt{3})(c + d\sqrt{3})$$

Show that $(\mathbb{Q}[\sqrt{3}], \oplus, \otimes)$ is a ring. (8 marks)