MACHAKOS UNIVERSITY

University Supplementary Examinations
SCHOOL OF PURE AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS AND STATISTICS
SMA 303: Ring Theory

DATE:

Question One - (30 marks)

(a) Explain the meaning of the following terms:

(i) a *non-commutative ring,* **(2 marks)**
(ii) an *integral domain,* **(2 marks)**
(iii) a *right ideal* in a ring R. **(2 marks)**

(b) Show that if R is a ring and $a, b, c, d \in R$ then $(a+b)(c+d) = ac + ad + bc + bd$. Justify your answer. **(4 marks)**

(c) Let $\mathbb{Q}[2]$ denote the set $\mathbb{Q}[2] = \{x + y\sqrt{2} : x, y \in \mathbb{Q}\}$. Show that the binary operation $\otimes$ defined by

$$\left(x + y\sqrt{2}\right) \otimes \left(z + w\sqrt{2}\right) = (xz + 2yw) + (xw + yz)\sqrt{2}$$

is an associative binary operation. **(4 marks)**

(d) Let R be a ring and $Z(R) = \{a \in R : ay = ya \; for \; all \; y \in R\}$ i.e. $Z(R)$ is the center of R. Prove that $Z(R)$ is a subring of R. **(4 marks)**

(e) Let $(\mathbb{Z}_{18}, \oplus, \otimes)$ denote the ring of integers modulo 18 under addition and multiplication modulo 18. Let I be the *principal ideal* of $\mathbb{Z}_{18}$ generated by 3.

(i) Determine all elements of I. **(4 marks)**
(ii) Construct all the elements of the *quotient ring* $\mathbb{Z}_{18}/I$. **(4 marks)**

(f) Let A be a ring with identity $f : A \to B$ be a surjective ring homomorphism. Prove that B is also a ring with identity. **(4 marks)**

Question Two - (20 Marks)

(a) Distinguish between a *zero divisor* and a *unit* in an ring R. **(4 marks)**

(b) Let R be a ring such that for any $a, b \in R$. Show that R is a commutative ring. **(5 marks)**

(c) Let $\mathbb{Z}[i] = \{x + iy | x, y \in \mathbb{Z}\} \subset \mathbb{C}$ be the set of Gaussian integers. Determine all the units in $\mathbb{Z}[i]$. **(5 marks)**

(d) Suppose that $\varphi : A \to B$ is a surjective ring homomorphism. Prove that

(i) if $I \subset A$ is a prime ideal then the image $\varphi(I)$ is a prime ideal of B, **(3 marks)**
(ii) if $J \subset B$ is a prime ideal then $\varphi^{-1}(J) = \{a \in A | \varphi(a) \in J\}$ is also a prime ideal.**(3 marks)**

Question Three - (20 Marks)

(a) Explain the meaning of the following terms.

(i) a *subring* S in a ring R, **(2 marks)**
(ii) a *principal ideal* in a ring R. **(2 marks)**

(b) Let $M_2(\mathbb{C})$ be the ring of all 2×2 matrices with complex entries. Consider the subset $T \subset M_2(\mathbb{C})$ given by

$$D = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$$

i.e. T is the subset containing upper triangular matrices. Prove that T is a subring of $M_2(\mathbb{C})$. **(5 marks)**

(c) Suppose that A and B are rings. Prove that if $\varphi: A \to B$ is a ring homomorphism then

(i) $\varphi(A)$ is a subring of B, **(4 marks)**
(ii) *Ker* φ is an ideal of A, **(4 marks)**
(iii) if A is a commutative ring then $\varphi(A)$ is commutative. **(3 marks)**

Question Four - (20 Marks)

(a) Explain the meaning of the following terms **(4 marks)**

(i) a *prime ideal* in a ring R,
(ii) a *maximal ideal* in a ring R.

(b) Suppose that U and V are ideals of a ring R. Prove that $U \cap V$ is also an ideal of R. **(4 marks)**

(c) Let $S = \{m + n\sqrt[3]{2} : m, n \in \mathbb{Z}\} \subset \mathbb{R}$. Determine whether S is a subring of $\mathbb{R}$. **(4 marks)**

(d) Let R be a ring and $f: \mathbb{Q} \to R$, $g: \mathbb{Q} \to R$ be ring homomorphisms such that $f(n) = g(n)$ for every $n \in Z$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Hint: Every element of $\mathbb{Q}$ can be written as $\frac{m}{n}$ for some $m, n \in Z$ with $n \neq 0$. **(8 marks)**

Question Five - (20 Marks)

(a) Consider the binary operations of addition and multiplication defined respectively on $\mathbb{R}^2$ by:

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \;\; and \; (x_1, x_2) \cdot (y_1, y_2) = (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1).$$

(i) Prove that $(R^2, +, \cdot)$ is a ring.**(6 marks)**
(ii) Consider the map $\psi: \mathbb{R}^2 \to \mathbb{R}$ defined by $\psi(x, y) = x + y$. Determine if ψ a ring homomorphism.**(4 marks)**

(b) Let $\mathbb{Q}[\sqrt{3}]$ denote the set $\{x + y\sqrt{3} : x, y \in \mathbb{Q}\}$. Define the binary operations $\oplus$ and $\otimes$ on $\mathbb{Q}[\sqrt{3}]$ as follows

$$(x + y\sqrt{3}) \oplus (z + w\sqrt{3}) \qquad (x + y) + (y + w)\sqrt{3}$$
$$(x + y\sqrt{3}) \otimes (z + w\sqrt{3}) \quad (xz + 3yw) + (xw + yz)\sqrt{3}$$

(i) Prove that $\mathbb{Q}[\sqrt{3}]$ is a subring of $\mathbb{R}$.**(4 marks)**

(ii) Determine if every non-zero elements of $\mathbb{Q}[\sqrt{3}]$ is invertible. **(6 marks)**