MACHAKOS UNIVERSITY

University Examinations 2019/2020 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

SMA 460: STOCHASTIC PROCESSES

QUESTION ONE (30 MARKS)

(a) Define terms as used in statistics:-

i) stochastic process **(1 marks)**

ii) continuous time stochastic process **(1 marks)**

iii) counting process **(1 marks)**

iv) periodic state **(2 marks)**

(b) Find the sample space for the experiment of tossing a coin,

(i) once **(1 marks)**

(ii) twice **(2 marks)**

(c) In an experiment of tossing a fair coin three times, the sample space, **S**, consists of eight equally likely sample points **S** = {HHH, …, TTT}. If X is the random variable giving the number of heads obtained find,

(i) P(X=2) **(3 marks)**

(ii) P(X<2) **(3 marks)**

(d) Let $\{X_i, i \geq 0\}$ be a simple random walk with $P(z_i=1) = p$ and $P(z_i = -1) = q = 1-p$ for all i, and let $X_i = \sum_{i=1}^{n} z_i$ $\quad i = 1, 2, \ldots$

and $X_0 = 0$

Now let the random process X(t) be defined by $X(t) = X_i$, $i \leq t < n+1$.

(i) Describe X(t) **(4 marks)**

(ii) Construct a typical sample function of X(t). **(5 marks)**

(e) Suppose the chance of rain tomorrow depends on the previous weather conditions only through whether or not it rains today and not on past weather conditions. Suppose also that if it rains today, it will rain tomorrow with probability α, and if it does not rain today then it will rain tomorrow with probability β.

(i) Describe the Markov chain. **(2 marks)**

(ii) Construct the matrix of transition probabilities. **(2 marks)**

(f) Given the matrix of transition probabilities

$$P = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Draw the state transition diagram. **(3 marks)**

QUESTION TWO (20 MARKS)

a) State three (3) conditions of a counting process. **(3 marks)**

b) Let $x_1, x_2, \ldots$ be independent Bernoulli random variables with $P(x_i=1) = p$ and $P(x_i=0) = q = 1-p$ for all i. The collection of random variables $\{X_i, i \geq 1\}$ is a random process, and it is called a Bernoulli process.

(i) Describe the Bernoulli process

(ii) Construct a typical sample sequence of the Bernoulli process and plot it.

(10 marks)

c) Let $z_1, z_2, \ldots$ be iid random variables with $P(z_i=1) = p$ and $P(z_i = -1) = q = 1-p$ for all i. Let

$$X_i = \sum_{i=1}^{n} z_i \qquad i = 1, 2, \ldots$$

and $X_0 = 0$

The collection of random variables $\{X_i, i \geq 0\}$ is a simple random walk X(i) in one dimension.

(i) Describe the simple random walk.

(ii) Construct a typical sample sequence (or realization) of X(i). **(7 marks)**

QUESTION THREE (20 MARKS)

(a) Define a Poisson process giving the conditions of a Poisson process. **(4 marks)**

(b) Patients arrive at a doctor's office according to a Poisson process with rate $\lambda = \frac{1}{10}$ minute.

The doctor will not see a patient until at least three patients are in the waiting room.

(i) Find the expected waiting time until the first patient is admitted to see the doctor.

(5 marks)

(ii) What is the probability that nobody is admitted to see the doctor in the first hour?

(6 marks)

(c) Suppose that people immigrate into an island at a Poisson rate $\lambda = 2$ per day.

(i) What is the expected time until the 20^{th} immigrant arrives? **(3 marks)**

(ii) What is the probability that the elapsed time between the 10^{th} and 11^{th} arrival exceeds 2 days? **(2 marks)**

QUESTION FOUR (20 MARKS)

(a) Define the terms

(i) Markov process

(ii) Markov chain **(4 marks)**

(b) Suppose that Rehema's mood tomorrow depends on whether or not she is happy today. If she is happy today then she will be happy tomorrow with probability 0.8, and if she is not happy today she will be happy tomorrow with probability 0.4.

(i) define the states of the Markov chain **(2 marks)**

(ii) construct the matrix of transition probabilities **(2 marks)**

(iii) calculate the probability that Rehema will be happy 2 days from today, and the probability that she will not be happy 4 days from today.

(7 marks)

(c) Consider the Markov chain consisting of the 4 states 0,1,2,3 and having transition probability matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(i) Determine the classes of this Markov chain. **(3 marks)**

(ii) Which is the absorbing state? Explain. **(2 marks)**

QUESTION FIVE (20 MARKS)

(a) Briefly describe the following terms as used in stochastic processes:-

(i) purely birth process

(ii) death process **(4 marks)**

(b) Define a Yule process. **(2 marks)**

(c) A birth-death process with 3 states, where the transition rate from state 2 to state 1 is $q_{21}=\mu$ and $q_{23}=\lambda$. Show that the mean time spent in state 2 is exponentially distributed with mean $\frac{1}{\lambda+\mu}$ **(7 marks)**

(d) Consider a Markovian system with discouraged job arrivals. Jobs arrive to a server in a Poisson fashion, with an intensity of one job per 7 seconds. The jobs observe the queue. They do NOT join the queue with probability l_k if they observe k jobs in the queue. l_k = k/4 if k<4, or 0 otherwise. The service time is exponentially distributed with mean time of 6 seconds.

(i) Determine the mean number of customers in the system, and **(4 marks)**

(ii) the number of jobs served in 100 seconds. **(3 marks)**