



MACHAKOS UNIVERSITY

University Examinations 2020/2021 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

THIRD YEAR SPECIAL /SUPPLEMENTARY EXAMINATION FOR
BACHELOR OF SCIENCE IN STATISTICS AND PROGRAMMING

BACHELOR OF SCIENCE IN MATHEMATICS

BACHELOR OF EDUCATION

SMA 300: REAL ANALYSIS I

DATE: 23/3/2021

TIME: 2.00-4.00 PM

INSTRUCTIONS

Answer question ONE and any other TWO questions

QUESTION ONE (30 MARKS) (COMPULSORY)

- (a) The set of real numbers $\mathbb{R}$ is the union of disjoint subsets $\mathbb{Q}$ and $\mathbb{Q}^c$ of Rational and Irrational numbers respectively.
- i) Giving examples, differentiate clearly between the two subsets. (4 marks)
 - ii) Given that a and b are rational with $b \neq 0$ and s is an irrational number such that $a - bs = t$. Show that t is irrational. (4 marks)
 - iii) Using (ii) above deduce that $\frac{\sqrt{5}-1}{\sqrt{5}+1}$ is irrational. (2 marks)
- (b) The following are basic concepts studied in Real analysis. For each concept give a definition and an example in $\mathbb{R}$.
- i) Neighborhood of a point. (2 marks)
 - ii) Interior point of a set. (2 marks)
 - iii) Open set. (2 marks)
 - iv) Limit point of a set. (2 marks)

- v) Closed set. (2 marks)
- (c) The following are subsets in $\mathbb{R}$
- (i) $P = (\frac{1}{6}, 3]$ (ii) $Q = \{1 + \frac{(-1)^n}{n}, n \in \mathbb{N}\}$ (iii) $S = \{y: y^2 \leq 9\}$.
- For each set examine boundedness and determine the following if they exist:
Infimum, Supremum, Maximum and Minimum values. (6 marks)
- (d) Prove that the union of arbitrary collection open sets is open. Exhibit an example. (4 marks)

QUESTION TWO (20 MARKS)

- (a) Find the largest ε such that A contains an ε -neighborhood of x_0 in each case:
- (i) $x_0 = \frac{3}{5}$, $A = [\frac{6}{7}, \frac{13}{11}]$ (ii) $x_0 = -0.722$, $A = (-1.35, \infty)$ (4 marks)
- (b) Define what is meant by a convergent sequence hence show from first principles that the sequence: $x_n = 8 + \frac{(-1)^n}{n^2} \forall n \in \mathbb{N}$ Converges to 8 in $\mathbb{R}$. (4 marks)
- (c) Give the definition of the concepts of limit superior and limit inferior of any sequence (x_n) of real numbers hence determine the convergence of the following sequence:
- $$x_n = \frac{(-1)^{n+1}}{5n-2} \forall n \in \mathbb{N} \quad (6 \text{ marks})$$
- (d) Show that a sequence $\{x_n\}$ of real numbers converges to x if and only if every subsequence of $\{x_n\}$ converges to x . (3 marks)
- (e) Show that $\lim_{n \rightarrow \infty} \left(\frac{3n+2}{n+1}\right) = 3$ using any appropriate technique. (3 marks)

QUESTION THREE (20 MARKS)

- (a) Let $A_n = \left(1 - \frac{1}{n}, 2 + \frac{1}{n}\right)$ $n \in \mathbb{N}$. Find:
- (i) $\bigcup_{n=1}^{\infty} A_n$ (ii) $\bigcap_{n=1}^{\infty} A_n$ (4 marks)
- (b) Show that a finite intersection of open sets is open. (4 marks)
- (c) Let A and B be neighborhoods of a point x . Show that the intersection, $A \cap B$ is a neighborhood of x . (4 marks)
- (d) Show that every open interval in $\mathbb{R}$ is a neighborhood of each of its points i.e. is an open set. (4 marks)
- (e) Show that if $r = \sqrt{m+1} - \sqrt{m-1} \forall m \in \mathbb{N}$, then r is irrational. (4 marks)

QUESTION FOUR (20 MARKS)

- (a) You are given the following sets.

$$A = \left\{ \frac{(-1)^n}{n}, n \in \mathbb{N} \right\}, \quad B = \left\{ 1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, \dots \right\}. \text{ Show that:}$$

- i) A is neither open nor closed. (3 marks)
 - ii) B is closed but not open. (3 marks)
- (b) Prove that:
- i) The union of any finite number of closed sets is closed. (3 marks)
 - ii) The intersection of an arbitrary collection of closed sets is closed. Exhibit an example. (4 marks)
- (c) Describe what is meant by statement “*The set $\mathbb{R}$ of real numbers is complete*”. Exhibit an example to illustrate why the set $\mathbb{Q}$ of rational numbers is not complete. (3 marks)
- (d) Let S be a bounded subset of the set of real numbers. Prove that $\sup S$ and $\inf S$ both belong to S if and only if S is closed. (4 marks)

QUESTION FIVE (20 MARKS)

- (a) Show that the set $\mathbb{Z}$ of integers is countable, hence conclude that the set of rational is also countable. (4 marks)
- (b) Define what is meant by a function f being uniformly continuous. (2 marks)
- (c) Let $f(x) = x^2 + 1 \forall x \in \mathbb{R}$. Prove that $\lim_{x \rightarrow \infty} f(x) = 5$. (3 marks)
- (d) Prove that the series $\sum_{n=1}^{\infty} \left[\frac{1}{n^p} \right]$ converges if $p < 1$ and diverges if $p \leq 1$. (4 marks)
- (e) Apply the indicated test of convergence:
- (i) $\sum \frac{\sqrt{n}}{n^2+1}$ (comparison test). (2 marks)
 - (ii) $\sum \frac{1}{n^n}$ (Root test). (2 marks)
 - (iii) $\sum \frac{1}{n^2+1}$ (Integral test). (3 marks)