MACHAKOS UNIVERSITY

University Examinations for 2020/2021

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR FIRST SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (STATISTICS AND PROGRAMMING)

BACHELOR OF EDUCATION (SCIENCE)

SMA 402: FIELD THEORY

DATE: 10/8/2021 **TIME: 8:30 – 10:30 AM**

INSTRUCTION: *Answer Question **ONE** which is compulsory and any other **TWO** Questions*

QUESTION ONE (COMPULSORY) (30 MARKS)

a) Let F, K be fields. Explain the meaning of the following terms:

(i) a *field extension* F/K, (2 marks)

(ii) an *algebraic element* $\alpha \in F$ over K, (2 marks)

(iii) an *algebraic field extension* F/K. (3 marks)

b) Let p be a prime number. Using relevant examples distinguish between a field of characteristic p and a field of characteristic 0. (5 marks)

c) Consider the polynomials $f(x) = 2x^5 + 3x^4 + 2$ and $g(x) = 3x^9 + 4x^4$ in $\mathbb{Z}_6[x]$.

(i) Compute the sum $f(x) + g(x)$. (3 marks)

(ii) Determine the degree of the product $f(x)g(x)$. (4 marks)

d) Let $p \in \mathbb{Q}$ be a square free rational number (e.g. a prime integer). Describe the field extension $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$. (6 marks)

e) Prove that the number $\sqrt[3]{2}$ is algebraic over $\mathbb{Q}$. (5 marks)

QUESTION TWO (20 MARKS)

f) Explain the meaning of the term *monic polynomial* and give an example of a monic polynomial in $\mathbb{R}[x]$. (2 marks)

g) Let F be any field. Prove that any polynomial $f(x) \in F[x]$ of degree 2 or 3 is irreducible if and only if $f(x)$ has no roots in F. (4 marks)

h) Using the root theorem, Show that (4 marks)

(i) $x - 1$ divides $x^{2021} + x^{2020} + x + 1$ in $\mathbb{Z}_2[x]$,

(ii) $x - 1$ divides $x^n - 1$ for any $n \geq 1$ in $\mathbb{R}[x]$.

i) Let $\alpha = 3\sqrt{2} + 3i \in \mathbb{C}$. Determine the minimal polynomial of α over $\mathbb{Q}$ hence or otherwise construct a basis for the field extension $\mathbb{Q}(\alpha)/Q$. (4 marks)

j) Suppose that F is a field extension of K and $\alpha \in F$ is algebraic over K. Let $f \in K[x]$ be a non-zero polynomial of the least degree such that $f(\alpha) = 0$. Prove that (6 marks)

(i) $f(x)$ is irreducible over K.

(ii) if $p(x) \in K[x]$ is a polynomial such that $p(\alpha) = 0$ then $f(x)|p(x)$.

QUESTION THREE (20 MARKS)

a) Distinguish between the term *algebraically closed field* K and that *algebraic closure* $\overline{K}$ of a field K. (3 marks)

b) Determine the root of $5x + 3$ in $\mathbb{Z}_7[x]$. (2 marks)

c) Determine whether the following statements are true or false. Justify your answers.

(4 marks)

i) $\mathbb{Q}[\sqrt{2}]$ is an algebraically closed field.

ii) $\mathbb{R}$ is an algebraically closed field.

d) Determine the minimal polynomial and the basis of the simple algebraic extension $\mathbb{Q}(\sqrt[3]{2} + 2\sqrt{5})$ over $\mathbb{Q}$. (6 marks)

e) Construct the splitting field of $f(x) = x^5 - 1$ over $\mathbb{Q}$. (5 marks)

QUESTION FOUR (20 MARKS)

a) Let $f(x) \in K[x]$ be a non-constant polynomial. Explain the meaning of the *splitting field* of $f(x)$ over K. (2 marks)

b) Determine whether the polynomial $x^2 + 2x + 4$ is irreducible in $\mathbb{Z}_5[x]$. (4 marks)

c) Let $K \subset L \subset F$ be fields. Show that if $\alpha \in F$ is algebraic over L then α is algebraic over K. (5 marks)

d) Determine the degree of the field extension $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ over each of the following fields (5 marks)

(i) $\mathbb{Q}$

(ii) $\mathbb{Q}(\sqrt{2})$

(iii) $\mathbb{Q}(\sqrt{6})$

e) Suppose that E/F is an extension satisfying $[E:F] = p$ where p is a prime number. Prove that E/F contains no proper intermediate field (i.e. there is no field L such that $F \subsetneq L \subsetneq E$). (4 marks)

QUESTION FIVE (20 MARKS)

a) Let $F[x]$ be a polynomial ring over a field F . Prove that $\alpha \in F[x]$ is invertible if and only if α is a non-zero element of F. (5 marks)

b) Suppose that K_1 and K_2 are subfields of a field K. (5 marks)

(i) Prove that $K_1 \cap K_2$ is a subfield of K.

(ii) Describe the meaning of a *prime* subfield of a field K.

c) Suppose that F/L and L/K are field extensions such that $[F:L] = n$ and $[L:K] = m$. Prove that F/K is an extension such that $[F:K] = nm$. (10 marks)