MACHAKOS UNIVERSITY

University Examinations 2018/2019

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE

SECOND YEAR SECOND SEMESTER EXAMINATIONS FOR

BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)

BACHELOR OF SCIENCE (STATISTICS AND PROGRAMMING)

BACHELOR OF SCIENCE (MATHEMATICS)

SST 102: DISCRETE MATHEMATICS

SMA 204: ALGEBRAIC STRUCTURES

DATE: 08/05/2019 **TIME: 2:00 – 4:00 PM**

INSTRUCTIONS

Answer Question One and any other two questions

QUESTION ONE (COMPULSORY) (30 MARKS)

a) Explain the meaning of the following terms

i) A binary relation on a set A. (2 marks)

ii) An injective function (2 marks)

b) Let $\mathbb{P}$ denote the set of strictly positive real numbers and let $\times$ be the usual multiplication of real numbers. Is $\langle\mathbb{P},\times\rangle$ a group? Justify your answer. (5 marks)

c) Let n be a fixed positive integer strictly greatest than 1. For any $x, y \in \mathbb{Z}$ then x is said to be congruent to y modulo n (written $x \equiv y \pmod{n}$)if $x - y$ is divisible by n. Show that this relation is *reflexive, symmetric* and *transitive*. (5 marks)

d) Consider the set $\mathbb{Z}_7 = \{0,1,2,3,4,5,6\}$ and let $\oplus$ denote operation of addition modulo 7 on the set $\mathbb{Z}_7$. Construct the Cayley table for the operation $\oplus$ on $\mathbb{Z}_7$. (5 marks)

e) Consider $\mathbb{Z}_{16}$ under multiplication modulo 16 (denoted by $\otimes$).

i) Is $\langle \mathbb{Z}_{16}, \otimes \rangle$ a group? Justify your answer. (3 marks)

ii) Determine the elements of U_{16}. (4 marks)

iii) Let H be the cyclic subgroup of U_{16} generated by $3 \in U_{16}$. Write down the elements of H. (4 marks)

QUESTION TWO (20 MARKS)

a) Let $G = \{x \in \mathbb{R}: x > 1\}$ be the set of all real numbers greater than 1. For $x, y \in G$ define $x \odot y = xy - x - y + 2$.

i) Show that the binary operation $\odot$ is closed on G. (2 marks)

ii) Show that the binary operation $\odot$ is associative on G. (2 marks)

iii) Show that 2 is the identity element for the binary operation $\odot$. (2 marks)

iv) Show that every element $g \in G$ has an inverse with respect to the binary operation $\odot$. (2 marks)

b) Let $\mathbb{Q}$ be the group of rational numbers under standard addition. Let Q_4 be the subset of $\mathbb{Q}$ given by

$$Q_4 = \left\{x \in \mathbb{Q} | x = \frac{m}{4} \text{ for some } m \in \mathbb{Z}\right\}.$$

Determine whether Q_4 is a subgroup of $\mathbb{Q}$. (5 marks)

c) Suppose that $(G, *)$ is a group and $a \in G$ be an element such that $a^2 = a$. Prove that $a = e$ where e is the identity element in G. (2 marks)

d) Let $M_2(\mathbb{Z})$ denote the group of all 2×2 matrices with integer entries under matrix addition. Let T be the subset of $M_2(\mathbb{Z})$ given by

$$T = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z}) \text{ such that } a, b, c \text{ and } d \text{ are multiples of } 3\right\}.$$

Show that T is a subgroup of $M_2(\mathbb{Z})$. (5 marks)

QUESTION THREE (20 MARKS)

a) Explain the meaning of the greatest common divisor of two integers. When do we say that two integers are relatively prime? (4 marks)

b) State the division algorithm for integers (without proof). (4 marks)

c) Use the division algorithm to write the greatest common divisor of $a = -77$ and $b = 56$ in the form $d = am + bn$ where m and n are integers. (6 marks)

d) State the Lagrange's theorem. (2 marks)

e) Suppose that G is a group of order 105 and suppose that H is a non-trivial subgroup of G. What are the possible values of $o(H)$? Justify your answer. (4 marks)

Question Four - (20 Marks)

a) Use the Euclidean algorithm to show that the integers $a = 94$ and $b = 47$ are relatively prime. (4 marks)

b) Let $G = \langle \mathbb{Z}, + \rangle$ be the group of integers under the standard addition. Show that the set

$$\begin{aligned} S &= \{7n \mid n \in \mathbb{Z}\} \\ &= \{\cdots, -14, -7, 0, 7, 14, \cdots\} \end{aligned}$$

is a subgroup of G. (6 marks)

c) Let G be a group. What do we mean by the term *center of a group* and show that the center of G denoted by $Z(G)$ is a subgroup of G. (6 marks)

d) Let G be a group such that $(ab)^2 = a^2b^2$ for all $a, b \in G$. Prove that G is a commutative group. (4 marks)

QUESTION FIVE (20 MARKS)

a) Explain the meaning of the following terms (4 marks)

i) Left coset of a subgroup H.

ii) Division ring

iii) Integral domain

iv) Characteristic of a ring

b) Let $U_{14} = \{1,3,5,9,11,13\}$ be the group of invertible elements in $\mathbb{Z}_{14}$ under multiplication modulo 14. Let H be the cyclic subgroup of U_{14} generated by 13. List all the distinct left cosets of H. (5 marks)

c) Let $\mathbb{Z}_n$ be the set of integer modulo n, let $\oplus$ denote the addition modulo n on $\mathbb{Z}_n$ and let $\otimes$ denote the multiplication modulo n on $\mathbb{Z}_n$. Does $\langle \mathbb{Z}_n, \oplus, \otimes \rangle$ satisfy all the axioms for it to be a ring? (5 marks)

d) Let $M_2(\mathbb{Z})$ denote the set of all 2×2 matrices with integer entries. Let $+$ denote the standard addition of matrices and $\times$ denote the standard matrix multiplication. Prove that $\langle M_2(\mathbb{Z}), +, \times \rangle$ is a ring. Is this ring a division ring? (6 marks)