MACHAKOS UNIVERSITY

University Examinations 2021/2022 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

FOURTH YEAR SECOND SEMESTER EXAMINATION FOR

BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)

BACHELOR OF SCIENCE (STATISTICS AND PROGRAMMING)

BACHELOR OF SCIENCE (MATHEMATICS)

BACHELOR OF EDUCATION (SCIENCE)

SMA 407: MEASURE AND INTEGRATION

DATE: 8/12/2021 **TIME: 11:00 – 1:00 PM**

INSTRUCTION:

Answer Question One and Any Other Two Questions

QUESTION ONE - (30 MARKS)

a) Let X be any set and $\mathfrak{X}$ be a σ-algebra in X.

i. Explain the meaning of the term *measure* on $(X, \mathfrak{X})$. (3 marks)

ii. Let $\mu: \mathfrak{X} \rightarrow \mathbb{R}_e$ be a map such that for each $A \in \mathfrak{X}$ then $\mu(A) = 0$. Determine whether μ is a measure of $(X, \mathfrak{X})$. (5 marks)

b) Let $(X, \mathfrak{X}, \mu)$ be a measure space. Show that if $A, B \in \mathfrak{X}$ are elements such that $A \subseteq B$

i. Prove that $\mu(A) \leq \mu(B)$. (3 marks)

ii. Show that $\mu(B) = \mu(A) + \mu(B \setminus A)$. (4 marks)

c) Denote by μ^* the Lebesgue outer measure on $\mathbb{R}$.

i. Show that $\mu^*(\{x\}) = 0$ for any point $x \in \mathbb{R}$. (5 marks)

ii. Prove that if $E \subset \mathbb{R}$ is such that $\mu^*(E) = 0$ then E is L-meausrable. (5 marks)

d) Suppose that $(X, \mathfrak{X})$ is a measure space and $f: X \rightarrow \mathbb{R}_e$ is an $\mathfrak{X}$-meaurable function. Prove that cf is $\mathfrak{X}$-meaurable function for any constant $c \in \mathbb{R}$. (5 marks)

QUESTION TWO (20 MARKS)

a) Let $(X, \mathfrak{X})$ be a measurable space and let $c \in \mathbb{R}$. Suppose that f, g are $\mathfrak{X}$-measurable functions. Prove that

i. $f + g$ is $\mathfrak{X}$-measurable (5 marks)

ii. f^2 is $\mathfrak{X}$-measurable (5 marks)

b) Suppose that $f_1, \dots, f_n$ is a sequence of $\mathfrak{X}$-measurable functions on X. Show that:

i. The function $f_1 \vee f_2 \vee \cdots \vee f_n$ is $\mathfrak{X}$-measurable, (5 marks)

ii. And that the function $f_1 \wedge f_2 \wedge \cdots \wedge f_n$ is $\mathfrak{X}$-measurable. (5 marks)

QUESTION THREE (20 MARKS)

a) Explain the meaning on the term *simple function* on a set X. (2 marks)

b) Let $(X, \mathfrak{X})$ be a measurable space and suppose that $\varphi, \psi \in M^+(X, \mathfrak{X})$ are simple functions. Prove that

i. $\int c\varphi d\mu = c \int \varphi d\mu$ for any non-negative constant $c \in \mathbb{R}$, (4 marks)

ii. $\int (\varphi + \psi) d\mu = \int \varphi d\mu + \int \psi d\mu$. (6 marks)

c) State and prove the monotone convergence theorem. (8 marks)

QUESTION FOUR (20 MARKS)

a) Let $(X, \mathfrak{X}, \mu)$ be a measure space and $f, g, h: X \to \mathbb{R}_e$ be functions. Suppose that $f < g$ μ-almost everywhere on X and that $g < h$ μ-almost everywhere on X. Prove that $f < h$ μ-almost everywhere on X. (4 marks)

b) Let $f, g \in M^+(X, \mathfrak{X})$ be functions such that $f \leq g$. Prove that $\int f d\mu \leq \int g d\mu$. (5 marks)

c) Suppose that $f \in M^+(X, \mathfrak{X})$ and $E, F \in \mathfrak{X}$ are measurable sets such that $E \subseteq F$. Prove that $\int_E f\, d\mu \leq \int_F f\, d\mu$. (5 marks)

d) State and prove Fatou's lemma. (6 marks)

QUESTION FIVE (20 MARKS)

a) Let X be a noo-empty set $\mathfrak{X}$ be a σ-algebra on X. Explain the meaning of *countable sub-additive* map on $\mathfrak{X}$. (5 marks)

b) Denote the Lebesgue outer measure by μ^*. Show that $\mu^*(\emptyset) = 0$. (3 marks)

c) Suppose that E is a countable subset of $\mathbb{R}$. Prove that the Lebesgue outer measure of E is zero. (i.e. $\mu^*(E) = 0$). (5 marks)

d) Show that if E is Lebesgue non-measurable then there exist a proper subset $A \subset E$ such that $0 < \mu^*(A) < \infty$. (7 marks)