MACHAKOS UNIVERSITY

University Examinations for 2024/2025 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCES

FOURTH YEAR FIRST SEMESTER EXAMINATION FOR DEGREE IN
BACHELOR OF SCIENCE (STATISTICS AND COMPUTER PROGRAMMING)
BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)
BACHELOR OF SCIENCE (ACTUARIAL SCIENCE)
BACHELOR OF EDUCATION (SCIENCE)

SMA 460: STOCHASTIC PROCESSES

DATE: 20TH JANUARY, 2025 **TIME: 8.30 AM – 10.30 AM**

INSTRUCTIONS:
Answer question ONE and any other TWO questions

QUESTION ONE (30 MARKS) , COMPULSORY

(a) Differentiate between

(b) a *random variable* and a *random experiment*. (2 marks)

(c) a *simple random walk* and a *symmetric random walk*. (2 marks)

(d) Briefly explain what is meant by a *Gaussian process* in stochastic processes. (2 marks)

(e) Consider the discrete-time random process $\{X(n), n \in \mathbb{Z}\}$ in which the $X(n)$'s are *iid* with CDF $F_{x(n)}(x) = F(x)$. Show that this is a *strict sense stationary (sss)* process. (3 marks)

(f) Given $G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + ...,$ then $a_n = \frac{G^{(n)}(0)}{n!}$. (5 marks)

(g) Find the probability that a random walk of 8 steps with probability $p = 0.7$ ends at

i. Position $x = 5$

ii. Position $x = -4$ (5 marks)

(h) Suppose that $X \sim Bernoulli(0,1)$. Using the probability generating function technique, show that $var(X) = pq$. (5 marks)

(i) On any day, Mutiso is either cheerful (C), so-so (S), or glum (G). If he is cheerful today then he will be C, S or G tomorrow with respective probabilities 0.5, 0.4, 0.1. If he is feeling so-so today, then he will be C, S or G tomorrow with probabilities 0.3, 0.4, 0.3. If he is glum today, then he will be C, S or G tomorrow with respective probabilities 0.2, 0.3, 0.5. Construct the matrix of transition probabilities. (3 marks)

(j) State Chapman-Kolmogorov theorem. (3 marks)

QUESTION TWO (20 MARKS)

a) A man either uses his ***car*** or takes a ***bus*** or a ***train*** to work each day. The TPM of the Markov chain with these three states 1 ***(car)***, 2 ***(bus)***, 3 ***(train)*** is

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

And the initial probability is $(0.7, 0.2, 0.1)$.

Calculate $P(X_2 = 3)$. (4 marks)

b) Suppose the probability that a student will visit the library tomorrow depends on whether or not he visits the library today. He will visit the library tomorrow with probability **α** if he visited the library today, and he will visit the library tomorrow with probability **β** if he did not visit it today.

i. Determine the matrix of transition probabilities. (2 marks)

ii. Letting the state be **o** when he visits the library and **1** when he does not, determine the limiting probabilities. (7 marks)

c) Consider the stochastic matrix

$$\mathrm{P} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(i) Draw the transition diagram. (3 marks)

(ii) Identify the various classes. (4 marks)

QUESTION THREE (20 MARKS)

(a) Define the following terms as used in stochastic processes.

i. Independent increments. (1 mark)

ii. Stationary increments. (2 marks)

(b) State three conditions that are necessary for a counting process to be called a Poisson process. (3 marks)

(c) During off peak hours, students arrive at an eatery at a Poisson rate $\lambda = 1$ in every 30 minutes. What is the probability that the elapsed time between the 3rd arrival and the 4th arrival exceeds 40 minutes? (3 marks)

(d) Small aircraft arrive at an airport according to a Poisson process at a rate of 6 per hour.

i. What is the probability that exactly 5 aircraft arrive in the first hour (after opening) and exactly 7 arrive in the hour after that? (3 marks)

ii. What is the probability that fewer than 5 small aircraft arrive in the first hour and at least 10 arrive in the hour after that? (4 marks)

iii. What is the probability that exactly 5 small aircraft arrive in the first hour and exactly 12 small aircraft arrive in the first 2 hours? (4 marks)

QUESTION FOUR (20 MARKS)

(a) Briefly explain what is meant a M/M/1 queueing system with balking. (2 marks)

(b) Suppose that customers arrive at a Poisson rate of one per every 12 minutes, and that the service time is exponential at a rate of one service per 8 minutes. What are L and W? (4 marks)

(c) Suppose that it costs $c\mu$ dollars per hour to provide service at a rate μ. Suppose also that we incur a gross profit of A dollars for each customer served. If the system has a capacity N, what service rate μ maximizes our total profit? (let N=2, λ =1, A = 10 , C = 1) 8 marks)

(d) Find the average amount of time a customer spends in the system for an $M/M/2$ system. (6 marks)

QUESTION FIVE (20 MARKS)

(a) Briefly explain what is meant by the term *birth process* as used in stochastic processes. (2 marks)

(b) Find the mean size $\mu(t)$ of the population in the death process at time t. Show that $\mu(t)$ satisfies the differential equation

$$\frac{d\mu(t)}{dt} = -\mu.\mu(t)$$

where $p_{n_0}(0) = 1$. (5 marks)

(c) Show that in the case of the birth and death process, and when $\lambda = \mu$, then

$$G(s,t) = \left[\frac{1+(\lambda z-1)(1-1)}{1+\lambda t(1-s)}\right]^{n_0}$$ (7 marks)

(d) A colony of bacteria grows without deaths with a birth rate which is time dependent. The rate declines exponentially with time according to $\lambda(t) = \alpha e^{-\beta t}$, where α and β are constants. Find the mean population size at time t given that the initial size of the colony is n_0. (6 marks)