MACHAKOS UNIVERSITY

University Examinations for 2024/2025 Academic Year

SCHOOL OF PURE AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCES

FOURTH YEAR FIRST SEMESTER EXAMINATION FOR DEGREE IN
BACHELOR OF SCIENCE (MATHEMATICS AND COMPUTER SCIENCE)
BACHELOR OF EDUCATION (SPECIAL NEEDS EDUCATION)
BACHELOR OF SCIENCE (MATHEMATICS)
BACHELOR OF EDUCATION (SCIENCE)
BACHELOR OF EDUCATION (ARTS)

SMA 431: DIFFERENTIAL GEOMETRY

DATE: 14TH JANUARY ,2025 **TIME: 2.00 PM – 4.00 PM**

INSTRUCTIONS:
Answer question ***ONE (Compulsory)*** *and* ***any other TWO*** *questions*

QUESTION ONE *(COMPULSORY)* (30 MARKS)

(a) Find the equation of the tangent line and normal plane to the curve **(5 marks)**

$\vec{X} = (1+t)\ \hat{e_1} - t^2\ \hat{e_2} + (1+t^3)\ \hat{e_3}$ At t=1

(b) Obtain the equation of the principal normal line and osculating plane to the curve
$\vec{X} = \cos t\ \ \hat{e_1} + \sin t\ \hat{e_2} + t^3\hat{e_3}$ at $t = \frac{\pi}{2}$ **(5 marks)**

(c) Given the vectors $\vec{a} = \hat{e_1} - \hat{e_2}$, $\vec{b} = 2\hat{e_2} - \hat{e_3}$ and $\vec{c} = 2\hat{e_1} - \hat{e_3}$, prove that they are linearly dependent **(5 marks)**

(d) Find the directional cosines of the vector $\vec{A} = 2\hat{e_1} - 2\hat{e_2} + \hat{e_3}$ **(5 marks)**

(e) Determine the unit vector normal to the plane containing the points P(0,1,1) , Q(1,0,-1) and R(1,-1,0) **(5 marks)**

(f) Compute the length of the arc on the curve $\vec{X} = e^t \cos t\ \hat{e_1} + e^t \sin t\ \hat{e_2} + e^t\ \hat{e_3}$, on $0 \leq t \leq \pi$ **(5 marks)**

QUESTION TWO (20 MARKS)

(a) Show that along a curve $\vec{x} = \vec{x}(s)$, then **(5 marks)**

$$[\vec{x}\ \vec{\dot{x}}\ \vec{\ddot{x}} = k^2\tau\ ]$$

(b) Prove that the curvature of the spherical indicatrix $\vec{x_1} = \hat{t}(s)$ of the tangent is given by $k_1^2 = \frac{k^2+\tau^2}{k^2}$ **(7 marks)**

(c) For the curvature given by $2z = 5x^2 + 4xy + 2y^2$, find the principal curvatures and the principal sections at the origin **(8 marks)**

QUESTION THREE (20 MARKS)

(a) Determine that at a point $\vec{x} = \vec{x}(t)$, where $k \neq 0$ the torsion is given by **(10 marks)**

$$\frac{[\ \vec{x}'\ \vec{x}''\ \ \vec{x}'''\ ]}{\left|\vec{x}'\ \times\ \vec{x}''\right|^2}$$

(b) Show that the representation is natural **(10 marks)**

$$\vec{x} = \frac{1}{2}\left[s + \sqrt{s^2+1}\right]\ \hat{e_1} + \frac{1}{2}\left[s + \sqrt{s^2+1}\right]\hat{e_2} + \frac{1}{2}\sqrt{2}\ ln\left[s + \sqrt{s^2+1}\right]\hat{e_3}$$

QUESTION FOUR (20 MARKS)

(a) Show that the curve generated by $\vec{X} = (-1 + \sin 2t \cos 3t)\ \hat{e_1} + (2 + \sin 2t \sin 3t)\hat{e_2} + (-3 + \cos 2t)\hat{e_3}$, lies on the sphere whose centre is $\vec{A} = -\hat{e_1} + 2\ \hat{e_2} - 3\ \hat{e_3}$ **(5 marks)**

(b) Prove that the length of the curve $\vec{X} = 2a\left[\sin^{-1} t + t\sqrt{1-t^2}\ \right]\hat{e_1} + 2at^2\ \hat{e_2} + 4at\ \hat{e_3}$

Between the points $t = t_1$ and $t = t_2$ is $4\sqrt{2a}\,(t_2 - t_1)$ **(6 marks)**

(c) If $\vec{X} = \vec{X}(t)$ is a representation of a curve of class ≥ 2, prove that **(9 marks)**

$$|\boldsymbol{k}| = \frac{|\vec{x}' \times \vec{x}''|}{|\vec{x}'|^3}$$

QUESTION FIVE (20 MARKS)

(a) Show that the surface $z = x^2 - y^2$ has asymptotic lines along the directions x=y and x=-y. **(3 marks)**

(b) A vector $\vec{v} = (1,0,0)$ is rotated by 90^0 about the z-axis. Use Rodrigue's formula to find the new coordinates of the vector. **(3 marks)**

(c) Show that the curvature of the involutes $\vec{x}^* = \vec{x} + (c - s)\,\hat{t}$ of $\vec{x} = \vec{x}(s)$ is given by

$$k^{*2} = \frac{k^2+\tau^2}{(c-s)^2\,k^2}$$ **(7 marks)**

(d) Determine the first fundamental form of the surface $\vec{x} = (u + v)\,\widehat{e_1} + (u - v)\,\widehat{e_2} + (uv)\,\widehat{e_3}$ **(7 marks)**